Falling into a Black Hole

Relativity and Astrophysics
Lecture 36
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Outline

- Principle of Extremal Aging
  - Conservation of Energy
- Energy in Curved Schwarzschild Geometry
  - Formulation
  - Measurement of Total Energy
  - Clock on a Shell
- Free-falling object
  - Shell view – velocity and energy
  - Crunch time
    - Over the edge
    - Timescales
    - Worldline view
- Homework: Due Friday, Dec. 4th
  - Problems 2-5 and 3-7 in Exploring Black Holes
  - You might want to look at problem 2-6 (but not required)

Principle of Extremal Aging & Energy

- Principle of Extremal Aging
  - The path a free object takes between two events in spacetime is the path for which the time lapse between these events, recorded on the object's wristwatch, is an extremum.
- Energy
  - The principle of Extremal Aging and the metric (spacetime interval) leads to the relativistic expression for energy

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A moving particle

Flashes occur at three fixed locations and times of flashes #1 and #3 are fixed. At what time does the particle pass location 2 and emit the second flash?

- Consider a free particle traveling in a straight line in space observed in an inertial frame
- The particle emits three flashes (#1, #2, and #3) – as shown above
  - The segments A and B need not be the same length.
  - Given fixed spatial positions for the events and fixed times for #1 and #3, when will flash #2 occur?
- Find intermediate time by demanding that the wristwatch (proper time) from #1 to #3 be an extremum (Principle of Extremal Aging)
  - This result leads to a conserved quantity, the energy of the particle

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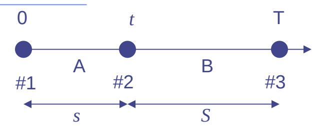
Event timing



Flashes occur at three fixed locations and times of flashes #1 and #3 are fixed. At what time does the particle pass location 2 and emit the second flash?

- Let t be the frame time between #1 and #2 and let s be the frame distances between the two flashes, The proper time is

$$\tau_A = \left(t^2 - s^2\right)^{1/2}$$

- Let T be the (fixed) frame time between flashes #1 and #3 and let S be the distance between them, so that the proper time in going from event #2 to #3 is

$$\tau_B = \left[(T-t)^2 - (S-s)^2\right]^{1/2}$$

- The total proper (wristwatch) time from event #1 to #3 is the sum of these two times

$$\boxed{\tau = \tau_A + \tau_B = \left(t^2 - s^2\right)^{1/2} + \left[(T-t)^2 - (S-s)^2\right]^{1/2}}$$

Applying Principle of Extremal Aging

- When – at what frame time t – will the stone, following its natural path, pass the intermediate point #2. We have so far

$$\boxed{\tau = \tau_A + \tau_B = \left(t^2 - s^2\right)^{1/2} + \left[(T-t)^2 - (S-s)^2\right]^{1/2}}$$

- The Principle of Extremal Aging can be used: the time t will make t an extremum.
  - To find the extremum we differentiate the above expression and set it to zero

$$\frac{d\tau}{dt} = \frac{t}{\left(t^2 - s^2\right)^{1/2}} + \frac{-(T-t)}{\left[(T-t)^2 - (S-s)^2\right]^{1/2}} = \frac{t}{\tau_A} - \frac{T-t}{\tau_B} = 0$$

  - Which tells us that

$$\frac{t}{\tau_A} = \frac{T-t}{\tau_B}$$

  - Defining t_A and t_B to be the times to travel the segments, we have

$$\boxed{\frac{t_A}{\tau_A} = \frac{t_B}{\tau_B}}$$

Constant of motion: Energy

- Now this holds for segments A and B but we didn't specify where these begin and end, thus these could be any consecutive segments, labeled A, B, C, D, … So that

$$\frac{t_A}{\tau_A} = \frac{t_B}{\tau_B} = \frac{t_C}{\tau_C} = \frac{t_D}{\tau_D} \ldots$$

- Thus we have for a free particle the ratio t/τ is a constant of the motion
- What is this quantity?

$$\frac{t}{\tau} = \frac{t}{\left(t^2 - s^2\right)^{1/2}} = \frac{t}{t\left(1 - (s/t)^2\right)^{1/2}} = \frac{1}{\left(1 - v^2\right)^{1/2}} = \frac{E}{m}$$

- Which just the energy per unit mass of a particle. It makes sense to use the instantaneous speed in case the particle changes speed, so

$$\frac{E}{m} = \frac{dt}{d\tau}$$

or in conventional units

$$\frac{E_{joules}}{m_{kg} c^2} = \frac{dt}{d\tau}$$

Energy in Curved Schwarzschild Geometry

- In the curved spacetime of Schwarzschild geometry the energy is given by

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

  - This can be derived in a manner similar to that for flat space (see pages 3-6 to 3-9 of textbook)
- Notes:
  - Particle of different mass follow the same worldlines (motion governed by energy per unit mass)
  - Using the Energy per unit mass has the advantage that it is "unitless"
- At large distances:

$$\frac{E}{m} = \frac{dt}{d\tau}$$

  - The energy for flat space, and $E \to m$ for an object at rest

Total Energy

- Unified energy
  - No separation between "kinetic" (speed) and "potential" (location) energy that appears in Newtonian mechanics.
- Can measure total energy of system (star plus satellite) via remote (Newtonian!) diagnostic probe

$$F = m_{probe} a = m_{probe} \frac{v^2}{r_{probe}} = \frac{GM_{total} m_{probe}}{r_{probe}^2} \quad \Rightarrow \quad M_{total} = \frac{v^2 r_{probe}}{G}$$

  - v^2/r is the acceleration of a circular orbit
- We then can get the energy of a satellite measured by a distant observer as

$$E = M_{total} - M_{star}$$

  - The value of the energy, E, associated with the satellite will be a constant of the motion (during free flight)

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Clock on a Shell

- Consider a clock of mass, m, bolted to a spherical shell at r-coordinate, r_o. What is the energy (with the remote probe)?

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

  - Ticks on the shell clock are the proper time (time read on the clock), so $d\tau = dt_{shell}$. From the Schwarzschild metric (previously)

$$dt = \left(1 - \frac{2M}{r}\right)^{-1/2} dt_{shell}$$

- So that

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)^{1/2} \le 1 \qquad \text{At rest at } r_o$$

- Thus the energy is less than the mass of the clock
  - This is the negative energy of gravitational binding
  - Note as $r_o \to 2M$, $E/m \to 0$ but a remote probe detects no change in the mass of the system

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Falling from Rest at Infinity

- Consider drop a object from a very large distance onto the black hole
- Energy is conserved
  - At a large distance for an object at rest, $E = m$ so that

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1$$

If the object were moving at large r, $E/m = 1/\left(1 - v_{far}^2\right)^{1/2}$

  - Will be a constant of the motion
- Squaring and using the Schwarzschild metric (for radial infall $d\phi = 0$) to substitute for $d\tau^2$ gives

$$\left(1 - \frac{2M}{r}\right)^2 dt^2 = d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2$$

- Solving for dr/dt

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}$$

Rate of change of r as measured on far-away clocks (bookkeeper)

  - The minus square root is used because radius decreases as an object falls

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Schwarzschild Bookkeeper view

- The bookkeeper derived velocity, dr/dt is

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r}\right)^{1/2}$$

Rate of change of r as measured on far-away clocks

- Note the “strange” (surprising) behavior of the particle
  - As it approaches the horizon ($r \to 2M$), $dr/dt \to 0$
  - The Schwarzschild bookkeeper which keeps track of the reduced circumference and far-away time reckons that the particle slows down as it approaches the horizon
  - The particle reaches the horizon only after infinite time
- Note
  - No one directly measures this speed
  - The remote observer can’t “see” the particle because of the “infinite” gravitational redshift at the horizon
- Let’s look at what a shell observer finds

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