1. Expansion rate, scale factor, and redshift of the universe.

Consider a reference frame whose origin is at the center of the Milky Way. We could just as well choose any other galaxy as the center because observations support the view that the universe is homogeneous and isotropic and that there is no preferred reference frame.

Let another galaxy have a coordinate distance $r$. Its distance is given by the product of $r$ and a scale factor $a(t)$,

$$\ell = a(t)r. \quad (1)$$

The distance $\ell$ has standard units of length (meters, etc.) so either $a$ or $r$ can have units of length with the other dimensionless. The scale factor depends only on cosmological time $t$. If it depended on location, the universe would not be homogeneous and it would not appear isotropic. To motivate this statement, consider three galaxies at comoving coordinates $(r_i, \theta_i, \phi_i)$ or $(x_i, y_i, z_i)$ with $i = 1, 2, 3$. Figure 1 demonstrates how a time-dependent scale factor can change distances but not the relative geometry of three galaxies. The comoving coordinates $r$ are fixed in time for galaxies that do not have 'peculiar' motions. Of course real galaxies do have such motions that are superposed with expansion.

Figure 1: A closed triangle shown at two epochs $t_{1,2}$. The scale factor $a(t)$ changes the size of the triangle but not its shape.

(a) Derive the Hubble law $v = H(t)\ell$ for an arbitrary epoch by using the definition of distances $\ell$ and letting $v$ be the velocity associated with the change in that distance per unit time. Relate the present-day Hubble constant $H_0 \equiv H(t_0)$ to the scale factor and its time derivative, $\dot{a} = da/dt$. 

1
(b) The scale factor stretches all lengths as the universe expands. Consider emission of light with rest wavelength $\lambda_0$, e.g. from a gas that is in a terrestrial lab. Now consider a distant object that emits the same spectral line at cosmological epoch $t < t_0$ that is measured with a telescope at time $t_0$. The size of the universe at these two epochs has the ratio $a(t)/a(t_0)$. Explain why the ratio of measured wavelength $\lambda$ to rest wavelength $\lambda_0$ is

\[ \frac{\lambda}{\lambda_0} = \frac{a(t_0)}{a(t)} . \]  
(2)

Note that the wavelength at the time of emission is $\lambda_0$ and it is longer when measured at a later epoch. Then using the definition of redshift $z = (\lambda - \lambda_0)/\lambda_0$ show that redshift is related to the scale factor as

\[ 1 + z = \frac{a(t_0)}{a(t)} . \]  
(3)

(c) The matter density $\rho_m$ of the universe scales as $\rho \propto 1/\text{volume}$. How does $\rho_m$ vary with the scale factor $a$ and with the red-shift factor $1 + z$?

(d) The cosmic microwave background (CMB) is blackbody (BB) radiation with a characteristic temperature today of $T_0 = 2.7$ K. A characteristic feature of BB radiation is that the peak of its spectrum is at a photon energy given by $h\nu \approx kT$ where $h$ is Planck’s constant, $k$ is Boltzmann’s constant, and $\nu$ is the frequency.\(^1\) In terms of the wavelength $\lambda = c/\nu$ an equivalent expression is $hc/\lambda = kT$. Using this fact along with the way that wavelength scales with redshift, show that $T(z)/T_0 = 1 + z$. What was the temperature of the radiation at the epoch when matter and radiation decoupled, corresponding to a redshift $z \approx 1100$?

(e) Another feature of BB radiation is that the density of energy in the radiation is $u_{\text{rad}} \propto T^4$. Using the relationship between mass and energy for matter, $E = mc^2$, show that the “effective” mass density\(^2\) of radiation scales as $\rho_r \propto T^4 \propto a^{-4} \propto (1 + z)^4$. Compare this expression to the one you got for the matter density $\rho_m$ and make a conclusion as to why the universe was radiation dominated in the distant past.

2. Equation of motion for the universe. Here we will take a Newtonian approach to expansion. First consider a sphere with mass $M(\ell)$ inside a radius $\ell$. For a constant matter density $\rho$

\[ M(\ell) = \frac{4\pi}{3} \ell^3 \rho . \]  
(4)

The potential energy of a test mass $m$ at the surface of the sphere is

\[ U = -\frac{GM(\ell)m}{\ell} . \]  
(5)

The kinetic energy of the mass is

\[ W = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{\ell}^2 \]  
(6)

where $\dot{\ell} = d\ell/dt$ is the outwards radial velocity of the mass. If you are not familiar with derivatives, just consider a quantity like $\dot{\ell}$ as the rate of change of a distance with time, which is a velocity.

\(^1\)See e.g. http://en.wikipedia.org/wiki/Black-body_radiation

\(^2\)The effective mass density of radiation is just the equivalent mass contained in the radiation energy. We can freely go back and forth between mass and energy using $E = mc^2$. 

2
(a) Calculate the total energy of the mass \( E = W + U \) and show that
\[
E = m \left( \frac{1}{2} \dot{\ell}^2 - \frac{4\pi}{3} G \ell^2 \rho \right). \tag{7}
\]

(b) Now use the comoving coordinate \( r \) and cosmological scale factor \( a \) to define \( \ell = a(t) r \) and show that
\[
E = \frac{1}{2} m r^2 \left( a^2 - \frac{8\pi}{3} G a^2 \rho \right). \tag{8}
\]
Also define \( k = -(2/mr^2)E \) (not the same as Boltzmann’s constant!) to get
\[
\dot{a}^2 - \frac{8\pi}{3} G a^2 \rho = -k. \tag{9}
\]

(c) The meaning of this equation is that for \( E > 0 \) or \( k = -1 \), the particle \( m \) has velocity larger than the escape velocity and is on a hyperbolic orbit. For \( E = 0 \) or \( k = 0 \) the particle has exactly the escape velocity and is on a parabolic orbit. For \( E < 0 \) (\( k = +1 \)) the particle is bound. Extrapolating our treatment to the entire universe, we would say that the three cases correspond, respectively, to an open, a marginally open, and a closed universe. For the \( k = 0 \) case solve for the density \( \rho \) and call this the ‘critical’ density. Solve for the critical density using the present day Hubble constant \( (H_0 \approx 70 \text{ km s}^{-1} \text{ Mpc}^{-1}) \) by using the relation you found in Problem 1 between the Hubble constant and the scale factor. Give your answer in grams per cubic centimeter.

3. A general relativistic (GR) treatment of a homogeneous and isotropic universe gives the same equation of motion! and sometimes is written in this form as one of the Friedmann equations\(^3\)
\[
\dot{a}^2 + k = \frac{8\pi G}{3} \rho a^2 \tag{10}
\]
In GR all forms of energy and mass contribute to the effective mass density \( \rho \) and can be expressed alternatively as an energy density \( \rho c^2 \). Recent work has indicated that the equation of motion needs to contain a term that acts like a cosmological constant \( \Lambda \), which means replacing \( \rho \) with \( \rho + \Lambda c^2 / 8\pi G \). For the \( k = 0 \) case (currently thought to apply to our universe), the equation of motion becomes
\[
\dot{a}^2 = \frac{8\pi G}{3} \rho a^2 + \frac{\Lambda c^2 a^2}{3}. \tag{11}
\]
As the universe expands the first term becomes negligible compared to the second because \( \rho \to 0 \), implying
\[
\dot{a}^2 = \frac{\Lambda c^2 a^2}{3}. \tag{12}
\]
If you are comfortable with calculus show that a solution to this equation is
\[
a(t) = a(t_0) e^{Ht} \quad \text{for } t > t_0, \tag{13}
\]
where \( H = c \sqrt{\Lambda/3} \) is the Hubble constant for this kind of cosmology. Otherwise you can try showing that if you substitute the solution Eq. 13 into Eq. 12 that you get the same quantity on both sides of the equation. What does the solution for \( a(t) \) imply for the expansion?

\(^3\)e.g. http://en.wikipedia.org/wiki/Friedmann_equations