Astronomy 6570

Physics of the Planets

Planetary & Satellite Orbital Resonances
Orbital Resonances

Two satellites orbiting a planet.

\[ n_1 = \frac{2\pi}{P_1} = \left( \frac{GM}{a_1^3} \right)^{\frac{1}{2}} \]

\[ n_2 = \frac{2\pi}{P_2} = \left( \frac{GM}{a_2^3} \right)^{\frac{1}{2}} < n_1 \]
A. Resonance Condition

Consider motion of #1 relative to #2:

\[ n_{rel} = n_1 - n_2 \]

or

\[ \frac{1}{P_{rel}} = \frac{1}{P_1} - \frac{1}{P_2} \]

→ satellites in conjunction at interval \( P_{rel} = \frac{2\pi}{n_{rel}} \)

What happens if \( P_{rel} = mP_1 \) with \( m \) an integer?

→ conjunctions occur exactly every \( m \) orbits of #1, at the same place in orbit #1

→ perturbations are the same every conjunction

→ effect of perturbations can grow large

\[
P_{rel} = mP_1 \quad n_{rel} = \frac{2\pi}{mP_1} = \frac{n_1}{m}
\]

\[ \therefore m(n_1 - n_2) = n_1 \]

\[ \therefore (m - 1)n_1 = mn_2 \]

i.e. \( n_1 = \frac{m}{m-1}n_2 \)

This is referred to as an \( m: m-1 \) resonance

<table>
<thead>
<tr>
<th>( m )</th>
<th>resonance</th>
<th>Examples</th>
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<tr>
<td>2</td>
<td>2:1</td>
<td>Io &amp; Europa</td>
</tr>
<tr>
<td>3</td>
<td>3:2</td>
<td>PSR 1257+12, Pluto</td>
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<tr>
<td>4</td>
<td>4:3</td>
<td>Titan &amp; Hyperion</td>
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</table>
B. Resonance Geometry

- Examine motion of Sat #1 relative to #2
- Assume initial conjunction occurs when #1 is at periapse of its elliptical orbit:

Assume $m = 3$

$\Rightarrow P_{rel} = 3P_1$

$\Rightarrow$ conjunctions occur every 3rd periapse of #1

Also, $(m - 1)n_1 = mn_2$

$\therefore (m - 1)(n_1 - n_2) = n_2$

or $(m - 1)n_{rel} = n_2$

i.e. $n_{rel} = \frac{n_2}{m - 1}$

or $P_{rel} = (m - 1)P_2$

$\Rightarrow$ conjunctions also occur every 2 orbits of #2.
C. Resonance Effects & Examples

1. Resonances provide stable orbits in which planets, satellites, or asteroids may be trapped, and protected against close encounters or large perturbations to the orbit. Examples:

<table>
<thead>
<tr>
<th>Resonance Ratio</th>
<th>Examples</th>
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<tr>
<td>2:1</td>
<td>Io – Europa, Europa - Ganymede, Mimas - Tethys, Enceladus - Dione</td>
</tr>
<tr>
<td>4:3</td>
<td>*Thule - Jupiter, Titan - Hyperion</td>
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2:1 resonances

2:1 resonances

3:2 resonances

4:3 resonances

2. Resonances can increase orbital eccentricity or inclination, which may lead to enhanced tidal energy dissipation. (See TIDES notes). Examples:

- Io: forced $e \rightarrow$ volcanism
- Miranda: melted interior?
- Europa: melted surface/$\text{H}_2\text{O}$ “lava”?  
- Enceladus: South polar plumes
2. Satellite Resonances

Consider the perturbation of an inner satellite on an eccentric orbit by a more massive outer satellite \( m_2 \) (e.g., Enceladus & Dione). The disturbing function for body 1 is

\[
\mathbb{R}_1 = Gm_2 \left( \frac{1}{\Delta} - \frac{r_1 \cos \phi}{r_2^2} \right)
\]

For zero inclinations, \( \Delta = \left( r_1^2 + r_2^2 - 2r_1r_2 \cos \phi \right)^{1/2} \)

and \( r_1 = \frac{a_1(1-e_1^2)}{1+e_1 \cos \nu_1} \), \( r_2 = \frac{a_2(1-e_2^2)}{1+e_2 \cos \nu_2} \), \( \phi = \nu_2 - \nu_1 \)

The indirect term arises from the displacement of \( M \) away from the center of mass due to the presence of \( m_2 \).

In general, the expansion of \( \mathbb{R}_1 \) in terms of orbital elements (via \( r_1, r_2, \nu_1 \) and \( \nu_2 \)) is messy.

It may be written as a Fourier-like series in the mean longitudes \( (\lambda), \tilde{\omega}, \) and \( \Omega \):

\[
\mathbb{R}_1 = \frac{Gm_2}{a_2} \sum \left( \frac{a_1}{a_2} \right)^l \mathcal{F}_{\text{imp}} \left( i_1 \right) \mathcal{F}_{\text{imp'}} \left( i_2 \right) G_{lpq} \left( e_1 \right) G_{lp'q'} \left( e_2 \right) \cos \Psi
\]

where \( \Psi_{\text{imp}p'q'q} = (l - 2p + q)\lambda_1 - (l - 2p' + q')\lambda_2 - q\tilde{\omega}_1 + q'\tilde{\omega}_2 - (l - 2p - m)\Omega_1 + (l - 2p' - m)\Omega_2 \)

and the sum is over
\[2 \leq l < \infty, \quad 0 \leq m \leq l, \quad 0 \leq p \leq l, \quad 0 \leq p' \leq l, \quad -\infty \leq q \leq \infty, \quad -\infty \leq q' \leq \infty.\]
Some simplification arises for small $e$ and/or $i$,
as $F_{\text{imp}}(i) \propto (\sin i)^{|l-2p-m|}$
and $G_{\text{epq}}(e) \propto e^{|q|}$
i.e., the leading powers of $\sin i$ and $e$ are equal (in absolute value) to the coefficients of $\Omega$ and $\tilde{\omega}$ in $\psi$.
Note that the sum of all the coefficients in $\psi$ is zero (known as d'Alembert's rule), which arises from rotational symmetry in the problem.
The sum of the coefficients of $\Omega$ and $\Omega'$ is even, i.e., 0, 2, 4, $\cdots$, $-2$, $\cdots$ $-4$, $\cdots$ (due to North - South symmetry).
From the above, and for small $e$ and $i$, the dominant terms in $\mathbb{R}_1$ have arguments
$$\psi = (m + q)\lambda_1 - (m + q')\lambda_2 - q\tilde{\omega}_1 - q'\tilde{\omega}_2$$
with $q$ and $q'$ equal to 0 or $\pm 1$, or more explicitly:
$$\begin{cases}
\psi_1 = m(\lambda_1 - \lambda_2) & \cdots \mathbb{R}_1 \text{ is independant of } e_1 \text{ and } e_2 \\
\psi_2 = (m \pm 1)\lambda_1 - m\lambda_2 \mp \tilde{\omega}_1 & \cdots \mathbb{R}_1 \propto e_1 \\
\psi_3 = m\lambda_1 - (m \pm 1)\lambda_2 \pm \tilde{\omega}_2 & \cdots \mathbb{R}_1 \propto e_2
\end{cases}$$
In general, all of these perturbing terms affect the motion of $m_1$, leading to a complex super-position of perturbations at different frequencies. Perturbations can be especially large if the frequency $\psi$ is unusually small, as the perturbing forces act in the same directions for a long term. This is what is meant by an orbit - orbit resonance.

Since $\dot{\lambda}_1 \simeq n_1$ and $\dot{\lambda}_2 \simeq n_2 < n_1$ we see that the above arguments can be resonant only if:

\[\dot{\psi}_1 \approx 0 \Rightarrow n_1 \approx n_2 \Rightarrow a_1 \approx a_2\]
\[\dot{\psi}_2 \approx 0 \Rightarrow n_1 \approx \frac{mn_2 - \dot{\omega}_1}{m-1} \approx \left(\frac{m}{m-1}\right)n_2\]
\[\dot{\psi}_3 \approx 0 \Rightarrow n_1 \approx \frac{(m+1)n_2 - \dot{\omega}_2}{m} \approx \left(\frac{m+1}{m}\right)n_2\]

The 1\textsuperscript{st} case can lead to resonance only for co-orbital satellites or asteroids (e.g., Trojans, or Janus & Epimetheus).

The 2\textsuperscript{nd} and 3\textsuperscript{rd} cases are referred to as first order resonances, since the coefficients of $\lambda_1$ and $\lambda_2$ differ by 1 and account for many of the observed asteroid, satellite, and ring resonances. We will look further at the 2\textsuperscript{nd} case, for which

\[\Re_1 = -\frac{Gm_2}{a_1} e_1 \alpha F_m(\alpha) \cos \psi_2, \text{ where } \alpha \equiv \frac{a_1}{a_2}.\]
Lindblad resonance equilibria as a function of distance from exact resonance ($\nu$) for a given resonance strength ($\beta$).

Figure 1: Solid line is the exact solution of equation 22. Broken lines are asymptotic solutions (equations 23 and 24). Solutions for $\epsilon > \epsilon_c$ and $\varphi = \pi$ are unstable.

Figure 2: Dotted lines are for $\bar{\nu} = \pm 10^{-4}$, dashed line is for $\bar{\nu} = \bar{\nu}_e$, and solid line is for $\bar{\nu} = 0$. Equilibrium points correspond to zero-crossings of $F(\epsilon)$. Symbols correspond to equilibrium points in figure 1.
(h, k) plots for varying distances from exact resonance, ν

Figure 3: h-k plot for $\nu = +10^{-4}$. Motion is anticlockwise.

Figure 4: h-k plot for $\nu = 0$. Motion is anticlockwise.

Figure 5: h-k plot for $\nu = \tilde{\nu}_c$. Motion is anticlockwise.

Figure 6: h-k plot for $\nu = -10^{-4}$. Motion is clockwise around the small-e equilibrium point otherwise.
## Solar System Resonances

Important orbital resonances in the solar system include: (all listed arguments librate)

<table>
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<th>Category</th>
<th>Equation</th>
<th>Description</th>
<th>Order</th>
</tr>
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<tr>
<td>Planets</td>
<td>$2\lambda_n - 3\lambda_p + \varpi_p$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asteroids</td>
<td>$\lambda - \lambda_j$</td>
<td>Trojans</td>
<td>$0^{th}$ order</td>
</tr>
<tr>
<td></td>
<td>$3\lambda - 4\lambda_j + \varpi$</td>
<td>Thule</td>
<td>$1^{st}$ order</td>
</tr>
<tr>
<td></td>
<td>$2\lambda - 3\lambda_j + \varpi$</td>
<td>Hilda</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda - 2\lambda_j + \varpi$</td>
<td>Griqua</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda - 3\lambda_j + 2\varpi$</td>
<td>Alinda</td>
<td>$2^{nd}$ order</td>
</tr>
<tr>
<td>Jovian Satellites</td>
<td>$\lambda_I - 2\lambda_E + \varpi_I$</td>
<td></td>
<td>$1^{st}$ order</td>
</tr>
<tr>
<td></td>
<td>$\lambda_I - 2\lambda_E + \varpi_E$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_E - 2\lambda_G + \varpi_E$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_I - 3\lambda_E + 2\lambda_G$</td>
<td></td>
<td>$0^{th}$ order (The LAPLACE RELATION)*</td>
</tr>
<tr>
<td>Saturnian satellites</td>
<td>$2\lambda_m - 4\lambda_T + \Omega_M + \Omega_T$</td>
<td></td>
<td>$2^{nd}$ order</td>
</tr>
<tr>
<td></td>
<td>$\lambda_E - 2\lambda_D + \varpi_E$</td>
<td></td>
<td>$1^{st}$ order</td>
</tr>
<tr>
<td></td>
<td>$3\lambda_{Ti} - 4\lambda_H + \varpi_H$</td>
<td></td>
<td></td>
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</tbody>
</table>

*This relation follows from the two above, and is an example of a “3-body resonance”. It is equal to 180°, and prevents all three satellites ever being in conjunction simultaneously.*
In addition, there are several important near-resonant arguments which circulate slowly:

<table>
<thead>
<tr>
<th>Planets</th>
<th>2λ_J − 5λ_s</th>
<th>...</th>
<th>The Great Inequality</th>
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<tbody>
<tr>
<td></td>
<td>λ_U − 2λ_N</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Uranian satellites</td>
<td>λ_A − 2λ_U</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2λ_T − 3λ_0</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>λ_M − 3λ_A + 2λ_U</td>
<td>...</td>
<td>cf. Laplace relation</td>
</tr>
</tbody>
</table>

The so-called “Nice model” of the early solar system presumes that at one time Jupiter and Saturn passed through a 2:1 resonance resulting in large perturbations which destabilized many asteroids and “ejected” Uranus and Neptune to their current orbits.
Pluto’s orbit, as seen in a reference frame rotating with Neptune.

The two planets are in a 2:3 outer Lindblad resonance (ie., $m = 2$).

Pluto’s argument of perihelion also librates around $90^\circ$, avoiding both nodal crossings.
Saturn’s e and the GREAT INEQUALITY

900-year period
Kirkwood gaps in the Asteroid Belt
Orbital resonances and Kuiper Belt structure
Major resonances in Saturn’s rings

Note: The outer boundaries of the A and B rings are determined by the Janus 7:6 and Mimas 2:1 resonances.
Resonances in the outer part of Saturn’s A ring