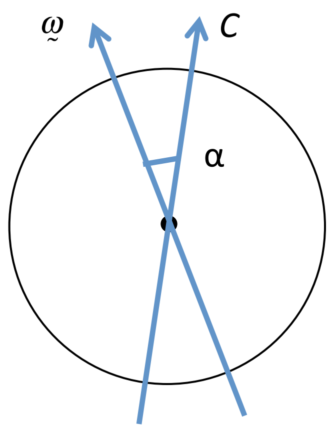
Astronomy 6570 – Physics of the Planets

Precession: Free and Forced

Planetary Precession

We have seen above how information concerning the distribution of density within a planet (in particular, the polar moment of inertia factor, C/MR^2) can be derived from measurements of the oblateness and J_2 of a rotating planet. In some cases, notably the Earth and Mars and probably Saturn in the relatively near future, we can also obtain such information from studying the planet's spin-axis precession. Precession may take one (or both) of two forms – free or Eulerian precession, and forced precession.

Free precession



Let the principal moments of inertia of the planet (or satellite) be $A \leq B < C$, and the corresponding body-fixed principal axes be denoted 1, 2, and 3. The angular momentum of the planet relative to its center of mass is

$$\underset{\sim}{H} = \underset{\approx}{I} \bullet \underset{\sim}{\omega}$$

$$= A\omega_1 \hat{x}_1 + B\omega_2 \hat{x}_2 + C\omega_3 \hat{x}_3$$ where $\underset{\approx}{I}$ is the inertia tensor and $\hat{x}_i$ are the principal axes.

If there are no external torques onthe planet, then we have

$$\frac{d\underset{\sim}{H}}{dt} = \frac{\partial \underset{\sim}{H}}{\partial t} + \underset{\sim}{\omega} \times \underset{\sim}{H} = 0$$

$$= \underset{\approx}{I} \bullet \dot{\underset{\sim}{\omega}} + \underset{\sim}{\omega} \times \left(\underset{\approx}{I} \bullet \underset{\sim}{\omega} \right)$$

Term due to rotating co-ordinate system

In terms of Cartesian components,

$$\text{A}\,\dot{\omega}_1 + \left(C - B\right)\omega_2\omega_3 = 0 \qquad (1)$$

$$\text{B}\,\dot{\omega}_2 + \left(A - C\right)\omega_3\omega_1 = 0 \qquad (2)$$

$$\text{C}\,\dot{\omega}_3 + \left(B - A\right)\omega_1\omega_2 = 0 \qquad (3)$$

These equations are know as Euler's equations. In the case of a planet flattened by rotation, so that A = B < C, they have a very simple solution:

$$(3) \;\Rightarrow C\,\dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{ constant}$$

$$\frac{d}{dt}(1): A\,\ddot{\omega}_1 + \left(C - A\right)\omega_3\dot{\omega}_2 = 0$$

$$\therefore A\,\ddot{\omega}_1 + \left(C - A\right)^2 \frac{\omega_3^2\omega_1}{A} = 0 \text{ from (2)}$$

$$\text{i.e. } \ddot{\omega}_1 + \left[\frac{C - A}{A}\omega_3\right]^2 \omega_1 = 0$$

Writing $\frac{C - A}{A}\omega_3 = \sigma,$ the general solution is

$$\omega_1 = \beta\cos\left[\sigma\left(t - t_0\right)\right]$$

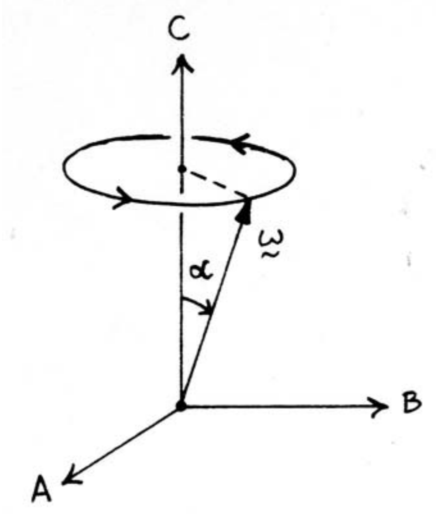
$$\therefore \omega_2 = +\frac{A}{C - A}\frac{\sigma\beta}{\omega_3}\sin\left[\sigma\left(t - t_0\right)\right]$$

$$= \beta\sin\left[\sigma\left(t - t_0\right)\right]$$

i.e., the instantaneous angular velocity vector, $\underset{\sim}{\omega}$, precesses around the C axis at a rate σ, with a constant angular displacement, α, given by

$$\tan\alpha = \frac{\beta}{\omega}$$

(See diagram next slide)



Such a precessional motion (known as the "**Chandler Wobble**") is observed for the Earth, with a very small amplitude of $\alpha = 0''.2 = 1\times10^{-6}$ rad. (The corresponding **linear** displacement of $\underset{\sim}{\omega}$ from the C axis at the Earth's poles is $\alpha R_{\oplus} \sim 6$ meters.)

The observed period, however, is $2\pi/\sigma_{obs} \sim 434$ days, whereas the period predicted by the above solution is

$$2\pi/\sigma_{obs} = \frac{A}{C-A} \bullet \frac{2\pi}{\omega_3} \simeq 306 \text{ days.}$$

(see below)

This discrepancy was unresolved for many years, but was eventually shown by Simon Newcomb to be due to non-rigidity of the Earth's mantle. This led to one of the earliest estimates of the Earth's elasticity by Lord Rayleigh*.

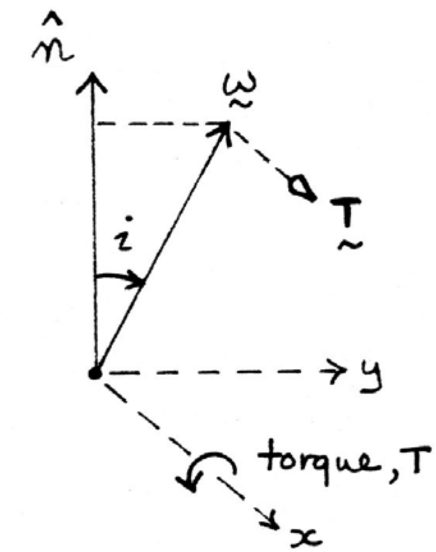
(Exercise left to the student: describe the motion, relative to **inertial** space, of the Earth's C axis during a precessional cycle.)

Footnote: Free precession is sometimes referred to as "Eulerian nutation" in mechanics books. This term is confusing, as the Earth also experiences an oscillatory component of its **forced** precession with an 18 year period which astronomers refer to as nutation.

*More recent studies have suggested that the excitation of the Chandler wobble is due to variations in salinity and temperature of the ocean, as well as changes in ocean currents and atmospheric circulation. Other possible contributors include large earthquakes.

Forced Precession

Because of the rotational flattening of a planet's figure, the sun and any large, non-equatorial satellites exert a torque on the planet which **attempts** to align the planet's spin axis with the normal to the orbit plane. The actual effect of such a torque is to force a precession of the spin axis about the orbit normal, as follows:



$\hat{n}$ = orbit normal

$\tilde{\omega}$ = spin vector

i = inclination of orbit on planet's equator

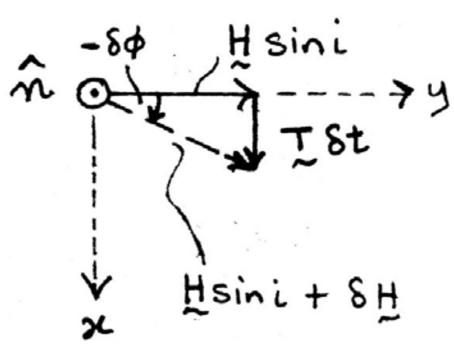
$\underset{\sim}{T}$ = satellite/solar torque

$\underset{\sim}{H} = C\underset{\sim}{\omega}$ = spin angular momentum

In time δt, the angular momentum changes by $\delta \underset{\sim}{H} = \underset{\sim}{T}\,\delta t$.

$\therefore$ the angular momentum vector rotates about n̂ by an angle

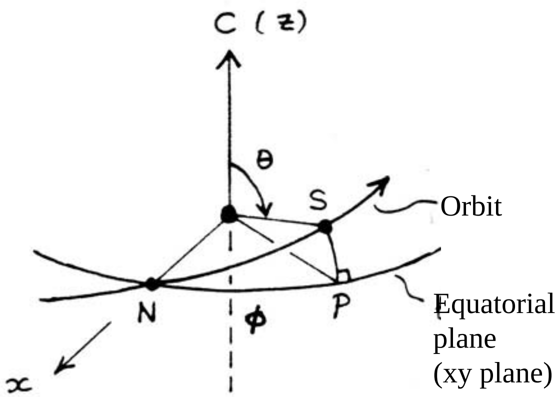
$$\delta\phi = -\frac{T\,\delta t}{H \sin i}$$



$$\Rightarrow \text{ precession rate } = \frac{d\phi}{dt} = -\frac{T}{H \sin i}$$

$$= -\frac{T}{C\omega \sin i}$$

To calculate T, we consider the reaction torque exerted **by** the planet **on** the satellite, averaged around one orbit:



S = satellite/sun

N = ascending node on Equator

$\angle$NP = ϕ

$\angle$SP = $\dfrac{\pi}{2} - \theta$

C = planet's spin axis

$\angle$NS = u

The planet's gravitational field is

$$V_G\left(r,\theta\right) \simeq -\frac{GM}{r} + \frac{GMR^2}{r^3} J_2 \; P_2\left(\cos\theta\right)$$

which leads to an instantaneous torque on S of

$$\boldsymbol{T} = \boldsymbol{r} \times \left(-m\nabla V_G\right), \text{ where m is the satellite mass}$$

$$= -m\frac{\partial V_G}{\partial\theta}\,\hat{\phi} = 3\frac{GMmR^2}{r^3} J_2 \sin\theta\cos\theta\,\hat{\phi}$$

Since $\boldsymbol{T}$ varies in both amplitude **and** direction as the satellite moves around its orbit, we resolve $\boldsymbol{T}$ into cartesian componnets T_x (towards N) and T_y, and average around one orbit:

$$\hat{\phi} = -\sin\phi\;\hat{x} + \cos\phi\;\hat{y}$$

$$\therefore\quad \left.\begin{aligned} T_x &= -T_0 \sin\theta\cos\theta\sin\phi \\ T_y &= T_0 \sin\theta\cos\theta\cos\phi \end{aligned}\right\} \text{ where } T_0 \equiv \frac{3GMmR^2}{r^3} J_2$$

From spherical trigonometry we derive the relations

$$\cos\theta = \sin i \sin u$$

$$\tan\phi = \cos i \tan u,$$

while r is given by the equation of the orbit:

$$r^{-1} = \frac{\left[1 + e\cos\left(u-\omega\right)\right]}{a\left(1-e^2\right)}.$$

We can simplify the algebra by assuming that

(i) $e \simeq 0$

(ii) $i << \dfrac{\pi}{2}$

so that we can set $r \simeq a, \phi \simeq u, \sin\theta \simeq 1,$ and $\cos\theta \simeq i \sin u.$

We then have the approximate results:

$$T_x \simeq -T_0 i \ \sin^2 u$$

$$T_y \simeq T_0 i \ \sin u \cos u$$

Upon averaging arounnd one orbit $\left(0 \le u \le 2\pi\right), T_y$ cancels and we have

$$\langle \boldsymbol{T} \rangle = -\frac{1}{2}\boldsymbol{T}_0 i \ \hat{x}$$

$$= -\frac{3}{2} GMm \frac{R^2}{r^3} J_2 i \ \hat{x}$$

A slightly more complicated analysis valid for all i yield

$$\langle \boldsymbol{T} \rangle = -\frac{3}{4} GMm \frac{R^2}{a^3} J_2 \sin 2i \ \hat{x},$$

showing that $\langle \boldsymbol{T} \rangle$ is zero for both $i = 0$ and $i = \pi/2$, and a maximum for $i = \pi/4$.

Returning to the precession rate formula, and noting that the torque exerted **by** the satellite **on** the planet is minus the above result, we have

$$\frac{d\phi}{dt}\Big/ planet = -\frac{3}{2}\frac{GMmR^2}{C\omega a^3} J_2 \cos i$$

This expression can be further simplified by substituting

$$J_2 = \frac{\left(C - A\right)}{MR^2}$$

and using Kepler's 3rd law: $n^2 a^3 = G\left(M + m\right)$:

$$\frac{d\phi}{dt} = -\frac{3}{2}\left(\frac{C - A}{C}\right)\left(\frac{m}{M + m}\right)\frac{n^2}{\omega}\cos i$$

The factor $\dfrac{m}{M + m}$ is $\sim \dfrac{m}{M}$ for satellite-induced precession, but ~1 for solar-induced precession.

Terrestrial forced precession

Let us evaluate the solar and lunar contributions to the Earth's precession rate:

	Sun	Moon
$\frac{m}{M+m}$	1.0	1/81.3
n	$2\pi/365^d$	$2\pi/27.3^d$
ω	$2\pi/1^d$	$2\pi/1^d$
$\frac{m}{M+m} \bullet \frac{n^2}{\omega}$	$4.72 \times 10^{-5} + 10.37 \times 10^{-5} = 15.1 \times 10^{-5}\ d^{-1}$	

So we see that the lunar term is dominant, and contributes ~ 69% of the total. The **observed** precession rate of the Earth's spin axis is

$$\frac{d\phi}{dt} = 50''.4 \text{ yr}^{-1} = 6.69 \times 10^{-7}\ d^{-1},$$

corresponding to a period of 25,600 yrs., and the inclination of **both** the sun's and moon's orbits to the equator is i ~ 23°.5, from which we may calculate the quantity

$$\frac{C-A}{C} = 0.00328 = 1/305$$

for the Earth.

This quantity may be combined with the measured value of

$$J_2 \equiv \frac{C - A}{MR^2} = 0.001083$$

To give the polar moment of inertia of the Earth:

$$C/_{MR^2} = 0.331$$

Note that this direct determination of C/MR^2 is in good agreement with that inferred indirectly from the Earth's rotational flattening using the Darwin-Radau approximation.

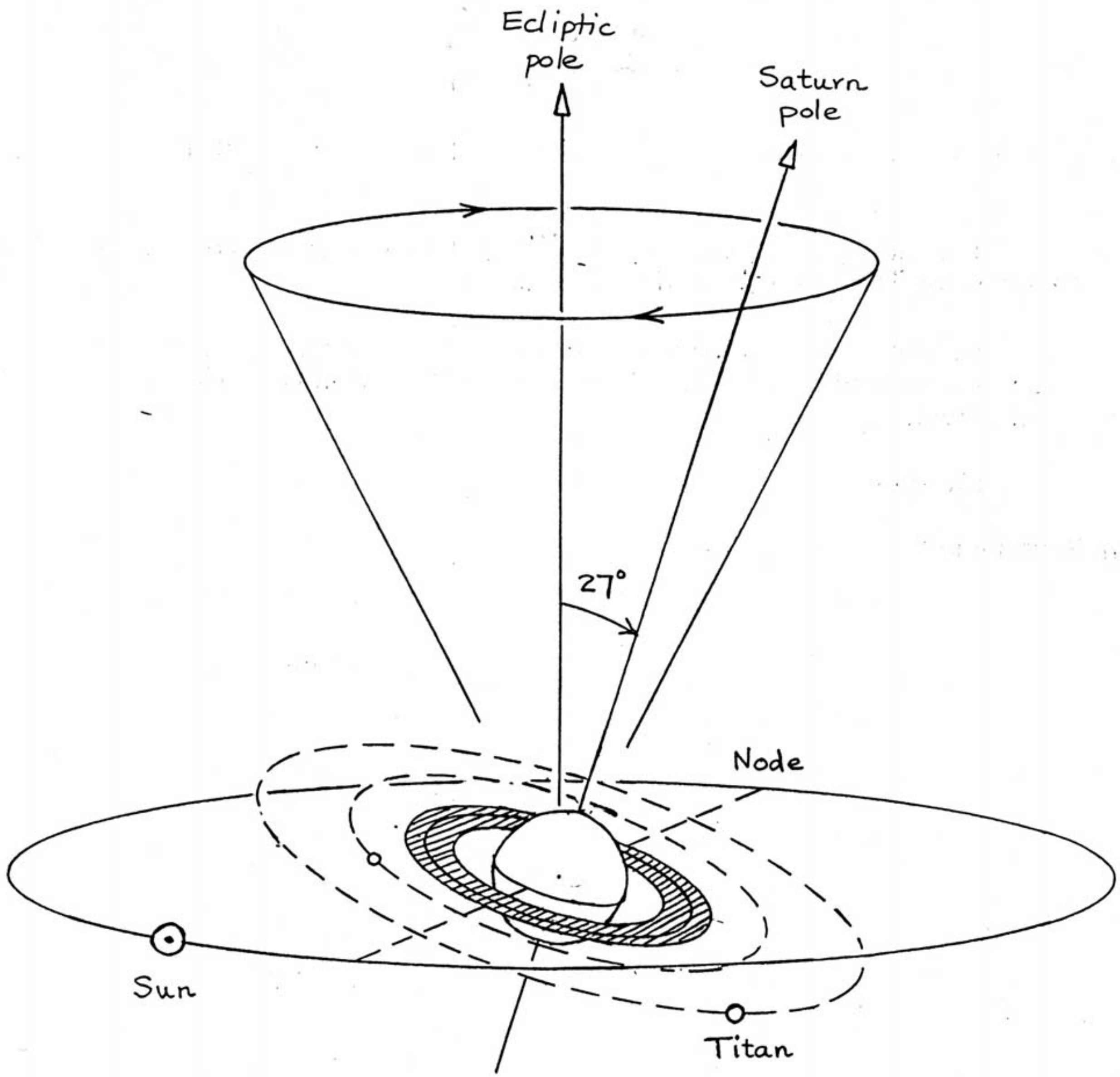
At present, no other planet but Mars has a measured forced precession rate (tracking of the 2 Viking landers on Mars was precise enough to do this), so we cannot generally apply this technique to determine accurate moments of inertia. In the future, however, such measurements may well be possible, at least Saturn.

Examples of spin precession periods.

Object	m	ε	T_{PRE}
Earth	Sun	23°.5	81,600 yr.
Earth	Sun + Moon	23°.5	25,700 yr.
Mars	Sun	25°.2	178,000 yr.
Jupiter	Sun	3°.1	500,000 yr.*
Saturn	Sun	26°.7	1,800,000 yr.*
Neptune	Sun	29°.4	23,000,000 yr.*
Moon	Earth	6°.7	78.5 yr.
Callisto	Jupiter	~0°.	200 yr.
Titan	Saturn	~0°.	200 yr.
Triton	Neptune	~0°.	65 yr.
Iapetus	Saturn	~9°	29,000 yr?

* Affected by solar torque on satellite orbits

Outer planet precession



We may estimate the actual forced precession rate of the Saturn system pole as follows. The torque exerted on Saturn's oblate figure by the sun, averaged over an orbit, is given by

$$T = -\frac{3GM_S M_\odot J_2 R_S^2}{4a_S^3} \sin 2\epsilon, \tag{4}$$

where M_S, R_S, ϵ and J_2 are the mass, equatorial radius, obliquity, and second zonal gravity coefficient of Saturn, respectively, $M_\odot$ is the mass of the sun, and a_S is the semimajor axis of Saturn's orbit. In terms of the planet's polar (C) and equatorial (A) moments of inertia,

$$J_2 = \frac{C - A}{M_S R_S^2}. \tag{5}$$

A similar torque is also exerted by the sun on the orbits of the equatorial satellites, which may conveniently be included in the above expression by adding the contribution of each satellite's orbit (treated here as a circular hoop of mass m_j and radius a_j) to the effective J_2 of the system:

$$J_2' = J_2 + \frac{1}{2} \sum_j \frac{m_j a_j^2}{M_S R_S^2}. \tag{6}$$

Titan is responsible for over 90% of the satellite torque.

These relatively weak sun-satellite torques are effectively communicated to the planet via the much stronger planet-satellite torques due to Saturn's oblateness, which lead to the satellites' maintaining their constant, near-zero inclinations to the planet's equatorial plane despite the steady precession of the planet's spin axis in inertial space (Goldreich 1965). More distant non-equatorial satellites such as Phoebe do not contribute appreciably to the effective torque on the system, as their orbits precess independently under solar perturbations, and thus do not maintain a fixed inclination to the planet's equator (*e.g.*, Burns 1977).

The precession period of the system is then determined by the augmented torque acting on the sum of Saturn's spin angular momentum and the combined orbital angular momenta of the equatorial satellites:

$$P = \frac{2\pi L_{\rm tot} \sin\epsilon}{T}, \tag{7}$$

where

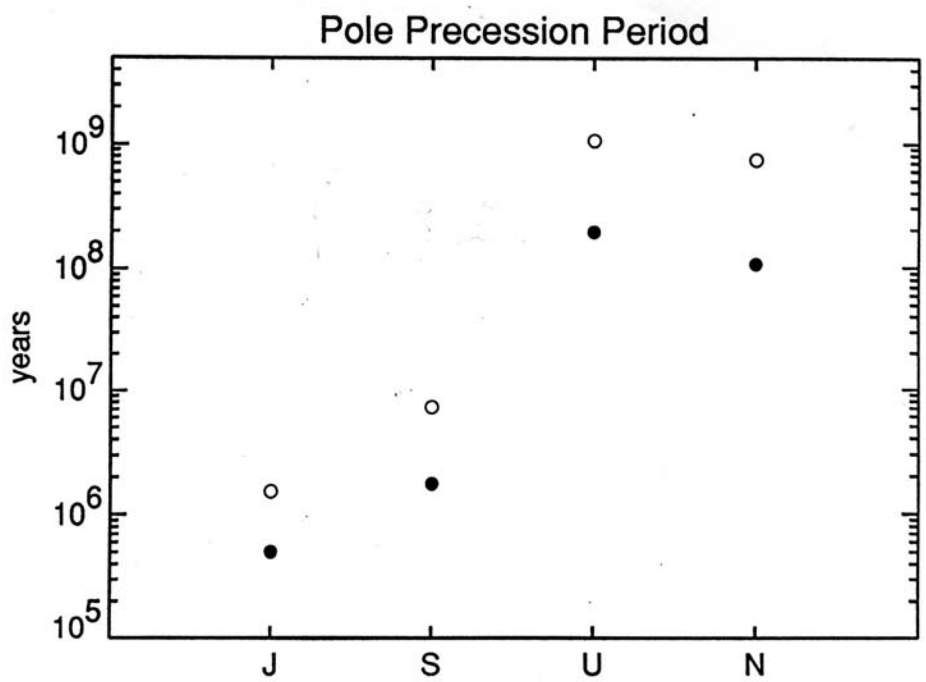
$$\begin{aligned} L_{\rm tot} &= C\omega + \sum_j m_j n_j a_j^2 \\ &= \gamma' M_S R_S^2 \omega. \end{aligned} \tag{8}$$

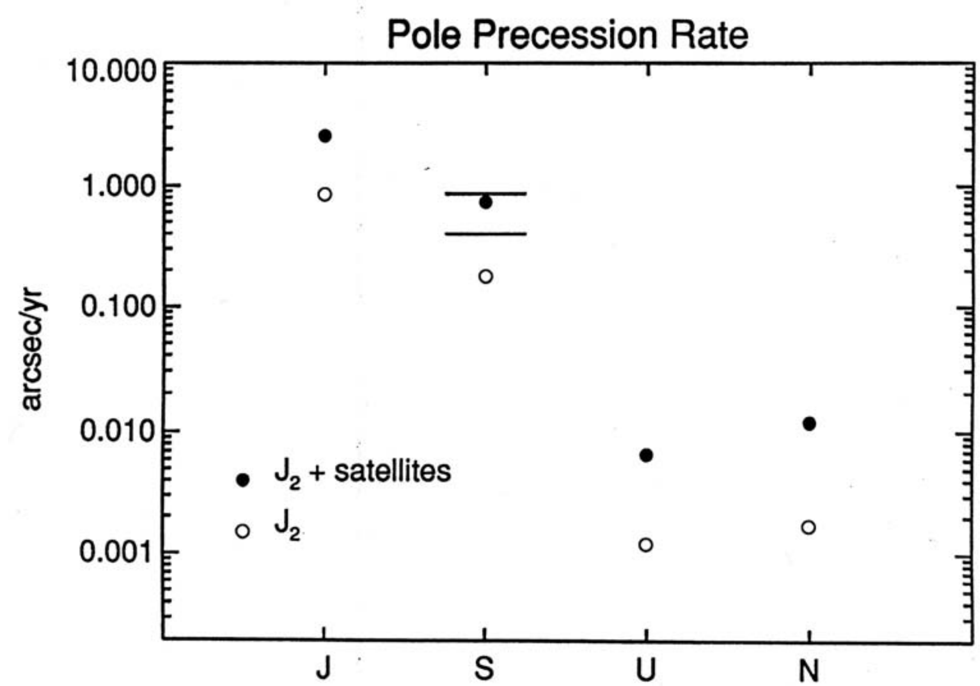
Here, ω is Saturn's spin angular velocity and the n_j are the satellite mean motions. The effective moment of inertia factor,

$$\gamma' = \gamma + \sum_j \frac{m_j a_j^2 n_j}{M_S R_S^2 \omega}, \tag{9}$$

where $\gamma = C/M_S R_S^2$. (The Iapetus contribution is reduced by a factor of $\sin(\epsilon - i_L)/\sin\epsilon$.) Combining the above expressions and using Kepler's third law, we have the more compact result (Ward 1975):

$$P = \frac{4\pi\gamma'\omega}{3J_2' n_S^2 \cos\epsilon}, \tag{10}$$


Pole Precession Period
years
10^9
10^8
10^7
10^6
10^5
J
S
U
N


Pole Precession Rate
arcsec/yr
10.000
1.000
0.100
0.010
0.001
J2 + satellites
J2
J
S
U
N

Digression: mutual precession

We have discussed (a) the nodal precession rate of satellite orbits due to the planetary J_2, and (b) the precession of the planet's spin axis due to the satellite torque on the equatorial bulge. How are these 2 different view points to be reconciled, and what if anything, remains fixed in space? The answer, of course, is that (for an isolated planet and satellite system) only the total annular momentum vector remains inertially fixed; both the spin axis of the planet and the orbit normal precess about this vector.

This is most easily shown using a vector notation for the torque, $\boldsymbol{T}$, exerted by the planet on the satellite;

$$\begin{aligned}\boldsymbol{T} &= -T_0 \sin i \cos i\, \hat{x} \\ &= -T_0 \left(\hat{s} \bullet \hat{n}\right) \hat{s} \times \hat{n}\end{aligned}$$

where $\hat{s}$ and $\hat{n}$ are unit vectors parallel to the planet's spin axis and the satellite's orbit normal, resp., and

$$T_0 = \frac{3G\left(C - A\right)m}{2a^3}.$$

Writing the spin and orbital angular momenta as

$$\boldsymbol{H} = H\,\hat{s} \text{ and } \boldsymbol{h} = h\,\hat{n}$$

we have the equations of motion:

$$\begin{aligned}\dot{\boldsymbol{H}} &= \dot{H}\,\hat{s} + H\,\dot{\hat{s}} = -\boldsymbol{T} = T_0\left(\hat{s} \bullet \hat{n}\right) \hat{s} \times \hat{n} \\ \dot{\boldsymbol{h}} &= \dot{h}\,\hat{n} + h\,\dot{\hat{n}} = \boldsymbol{T} = -T_0\left(\hat{s} \bullet \hat{n}\right) \hat{s} \times \hat{n}\end{aligned}$$

Now $\hat{s}$ and n̂ are unit vectors, so $\dot{\hat{s}} \perp \hat{s}$ and $\dot{\hat{n}} \perp \hat{n}$, and the right-hand sides of both equations are $\perp \hat{n}$ and $\perp \hat{s}$, so we must have

$$\dot{H} = 0 \text{ and } \dot{h} = 0.$$

i.e., the magnitudes of $\boldsymbol{H}$ and $\boldsymbol{h}$ remain constant. Furthermore, the angle i between $\boldsymbol{H}$ and $\boldsymbol{h}$ is given by

$$H\,h\cos i = \boldsymbol{H} \bullet \boldsymbol{h}$$

or

$$\cos i = \hat{s} \bullet \hat{n}$$

so

$$\frac{d}{dt}\left(\cos i\right) = \dot{\hat{s}} \bullet \hat{n} + \hat{s} \bullet \dot{\hat{n}} = 0$$

since $\dot{\hat{s}} \sim \hat{s} \times \hat{n} \perp \hat{n}$ and similarly for $\dot{\hat{n}}$ is $\perp$ ŝ. Thus the inclination remains constant also.

Finally, we look at the orientation of the plane defined by $\boldsymbol{H}$ and $\boldsymbol{h}$ and whose normal is given by $\hat{s} \times \hat{n}$

$$\frac{d}{dt}\left(\hat{s} \times \hat{n}\right) = \dot{\hat{s}} \times \hat{n} + \hat{s} \times \dot{\hat{n}}$$

$$= T_0\left(\hat{s} \times \hat{n}\right)\left\{\left(\hat{s} \times \hat{n}\right) \times \frac{\hat{n}}{H} - \hat{s} \times \frac{\left(\hat{s} \times \hat{n}\right)}{h}\right\}$$

$$= \frac{T_0\left(\hat{s} \times \hat{n}\right)}{Hh}\left\{\left(\hat{s} \times \hat{n}\right) \times \left(h\hat{n} + H\hat{s}\right)\right\}$$

$$= \frac{T_0 \cos i}{Hh}\left(\hat{s} \times \hat{n}\right) \times \boldsymbol{H}_T$$

where $\boldsymbol{H}_T$ is the (fixed) total angular momentum vector. Thus the vector $\left(\hat{s} \times \hat{n}\right)$ precesses around $\boldsymbol{H}_T$ at an angular rate

$$\frac{d\phi}{dt} = -\frac{T_0 H_T}{Hh}\cos i.$$

We can readily verify that this general expression reduces to our previous results in the limiting cases $h << H$ and H << h:

(i) **small satellite**, $h << H$:

$$H_T \simeq H, \text{ so } \frac{d\phi}{dt} \simeq -\frac{T_0}{h}\cos i$$

Now $$h \simeq (GM\,a)^{\frac{1}{2}}\, m \text{ for } e << 1,\, m << M$$

so $$\left.\frac{d\phi}{dt}\right| satellite \simeq -\,{}^{3}\!/_{2}\,\frac{G(C-A)m\cos i}{(GM)^{\frac{1}{2}}\, a^{\frac{7}{2}} m}$$

$$= -\,{}^{3}\!/_{2}\,(GM)^{\frac{1}{2}}\, J_2\, R^2 \cos i\; a^{-7/2}$$

$$= -\,{}^{3}\!/_{2}\, n\, J_2 \left({}^{R}\!/_{a}\right)^2 \cos i$$

$\cdots$ as before, for $e << 1$.

(ii) **large satellite**, h >> H :

$$H_T \sim h, \text{ so } \frac{d\phi}{dt} = -\frac{T_0}{H}\cos i$$

$$H = C\omega$$

$$\therefore \quad \left.\frac{d\phi}{dt}\right| planet \simeq -\,{}^{3}\!/_{2}\,\frac{G(C-A)m\cos i}{C\omega\, a^3}$$

$$= -\,{}^{3}\!/_{2}\,\frac{GM\; m\, R^2}{C\omega\, a^3}\, J_2 \cos i$$

$\cdots$ as before.

*Almost all satellites fall in case (i), except for Earth's moon which satisfies case (ii), and possibly Neptune's Triton, which may be an intermediate case. Case (ii) also applies to solar torques exerted on planetary spin vectors, and to planetary torques exerted on satellite spin vectors.

Summary of precession rate formulae

Orbital precession:

$$\dot{\tilde{\omega}} \simeq n_0 \left\{ \tfrac{3}{2} J_2 \left(R/a \right)^2 - \tfrac{15}{4} J_4 \left(R/a \right)^4 + \cdots \right\}$$

$$\dot{\Omega} \simeq -n_0 \left\{ \tfrac{3}{2} J_2 \left(R/a \right)^2 - \tfrac{9}{4} J_2^2 - \tfrac{15}{4} J_4 \left(R/a \right)^4 + \cdots \right\}$$

$\cdots$ due to **planet**

$$\dot{\tilde{\omega}} \simeq \tfrac{1}{4} n_0 \left(\frac{m_s}{M} \right) \alpha^2 b_{3/2}^{(1)}(\alpha) \qquad \left(\alpha \equiv a/a_s \right)$$

$$\dot{\Omega} \simeq -\dot{\tilde{\omega}}$$

$\cdots$ due to **exterior satellite**

$$\dot{\tilde{\omega}} \simeq \tfrac{1}{4} n_0 \left(\frac{m_s}{M} \right) \alpha b_{3/2}^{(1)}(\alpha) \qquad \left(\alpha \equiv a_s/a \right)$$

$\cdots$ due to **interior satellite**

$$\left.\begin{aligned} \dot{\tilde{\omega}} &\simeq \tfrac{3}{4} \quad n_\odot^2/n \left(1 - 2\sin^2\beta \right) \\ \dot{\Omega} &\simeq -\tfrac{3}{4} \quad n_\odot^2/n \cos\beta \end{aligned}\right\} n_\odot = \text{ planet's mean motion} \qquad \beta = \text{obliquity}$$

$\cdots$ due to the **sun**

Note : $b_{3/2}^{(1)}(\alpha) \simeq 3\alpha + \frac{45}{8}\alpha^3 + 0\left(\alpha^5\right) \qquad \cdots \alpha << 1$

$$n_0 \equiv \left(GM/a^3 \right)^{\frac{1}{2}}$$

$$n \simeq n_0 \left\{ 1 + \tfrac{3}{4} J_2 \left(R/a \right)^2 + \cdots \right\}$$

Spin axis precession:

Free precession:

$$\sigma = \frac{C-A}{A}\omega$$

Forced precession, due to mass '*m*' at distance '*a*'

$$\Omega = -\tfrac{3}{2}\left(\frac{C-A}{C}\right)\frac{Gm}{a^3}\,\omega^{-1}\cos\beta$$

$$= \begin{cases} -\tfrac{3}{2}\,\dfrac{C-A}{C}\,\dfrac{n_\odot^2}{\omega}\cos\beta & \cdots \text{ due to } \textbf{Sun} \\ -\tfrac{3}{2}\,\dfrac{C-A}{C}\,\dfrac{m}{M}\,\dfrac{n_2}{\omega}\cos i & \cdots \text{ due to } \textbf{Satellite} \end{cases}$$