Astronomy 570
Physics of the Planets

Thermal Emission and Blackbody Radiation
Black-body radiation

The radiance with wavenumber $\nu$, emitted by a black-body with temperature $T$, is described by the Planck function $B_{\nu}$:

$$B_{\nu}(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \text{ Wm}^{-2}\text{sr}^{-1}\text{cm} \quad (1)$$

with:

- $T$ the temperature
- $h$ Planck’s constant
- $\nu$ the frequency
- $c$ the speed of light
- $k$ Boltzmann’s constant

In wavelengths $\lambda$:

$$B_{\lambda}(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \text{ Wm}^{-2}\text{sr}^{-1}\text{m}^{-1} \quad (2)$$

(use: $\nu = c/\lambda$, $d\nu = -c/\lambda^2d\lambda$, and $B_{\nu}d\nu = -B_{\lambda}d\lambda$)
Wien's displacement law:

For a given $T$, $B_{\nu}$ is maximum at:

$$\nu_{\text{max}} \approx 2.0 \, T \, \text{cm}^{-1}$$  \hspace{1cm} (3)

thus, $B_{\nu_{\text{max}}} \sim T^3$

$B_{\lambda}$ is maximum at:

$$\lambda_{\text{max}} \approx 0.3 / T \, \text{cm}$$  \hspace{1cm} (4)

thus, $B_{\lambda_{\text{max}}} \sim T^5$
Rayleigh-Jeans approximation:

When $\nu \ll \nu_{\text{max}}$:

$$B_\nu(T) \approx \frac{2\nu^2 kT}{c^2} = \frac{2kT}{\lambda^2} \quad (5)$$

When $\lambda \gg \lambda_{\text{max}}$:

$$B_\lambda(T) \approx \frac{2ckT}{\lambda^4} \quad (6)$$

(use: $e^x \approx 1 + x + \ldots$ for $x \ll 1$)
Black-body flux or irradiance:

A black-body emits radiation isotropically, so the emitted flux at wavelength $\lambda$ is:

$$F_\lambda(T) = \int B_\lambda(T') \cos \theta d\Omega$$

$$= \int_0^{2\pi} \int_0^1 B_\lambda(T') d\phi \cos \theta d\cos \theta$$

$$= \pi B_\lambda(T') \text{ Wm}^{-2}\text{m}^{-1} \quad (7)$$

with $\Omega$ the solid angle ($d\Omega = d\phi d\cos \theta$)
Stefan-Boltzmann’s law:

The flux emitted in the whole black-body spectrum is then given by:

\[ F(T) = \pi \int_0^\infty B_\lambda(T) d\lambda \]  \hspace{1cm} (8)

Writing \( x = hc/\lambda kT \), this transforms into:

\[
F(T) = \frac{2\pi k^4 T^4}{h^3 c^2} \int_0^\infty \frac{x^3}{e^x - 1} dx \\
= \frac{2\pi^5 k^4}{15 h^3 c^2} T^4 \\
\equiv \sigma T^4 \text{ Wm}^{-2}
\]  \hspace{1cm} (9)

with \( \sigma \) the constant of Stefan-Boltzmann

\[ \sigma \approx 5.67 \cdot 10^{-8} \text{ Wm}^{-2}K^{-4} \]
Effective temperature $T_e$:

The total power emitted by an isothermal, black-body sphere with temperature $T$ and radius $R$, is thus:

$$P = 4\pi R^2 \sigma T^4 \equiv 4\pi R^2 \sigma T_e^4 \quad \text{W}$$

(10)

with $T_e$ the effective temperature.

For the sun, $T_e \approx 5800 \, \text{K}$:

The total power emitted by the sun is also denoted by $L_{\odot}$. 
Equilibrium temperature $T_{eq}$:

An estimate for the temperature of a planet with radius $R$, at a distance $r$ from the sun, can be made by equating the solar radiation it absorbs to the thermal radiation it emits, assuming the planet radiates as a black-body:

$$(1 - A)\pi R^2 \frac{L_\odot}{4\pi r^2} \equiv 4\pi R^2 \sigma T_{eq}^4 \quad (27)$$

Here, $T_{eq}$ is called the *equilibrium temperature*.

Writing:

$$S_\odot = \frac{L_\odot}{4\pi r_\odot^2} \quad (28)$$

with $r_\odot$ the distance between the earth and the sun, and $S_\odot$ the extraterrestrial flux, we get:

$$T_{eq} = \left[\frac{(1 - A)S_\odot}{4\sigma(r/r_\odot)^2}\right]^{1/4} \approx 278.4 \left[\frac{(1 - A)}{(r/r_\odot)^2}\right]^{1/4} \text{ K} \quad (29)$$
One can refine Equation 33 by introducing a factor $\alpha$ to describe the way the planet emits its thermal radiation:

$\alpha = 1$ sphere with uniform temperature
$\alpha = 1/2$ hemisphere with uniform temperature
$\alpha = 1/4$ flat surface normal to incident sunlight

Then:

$$T_{eq} \approx 278.4 \left[ \frac{(1 - A)}{\alpha(r/r_\oplus)^2} \right]^{1/4} \text{ K} \quad (30)$$
Differences between real $T$ and $T_{eq}$ are due to:

internal heat sources:
   Jupiter, Saturn, Neptune

greenhouse effect:
   Venus, Earth, Mars

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<th>$\bar{T}$</th>
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Figure 1.3 Atmospheric absorptions. (a) Black-body curves for 6000 K and 250 K. (b) Atmospheric absorption spectrum for a solar beam reaching the ground. (c) The same for a beam reaching the temperate tropopause. The areas beneath the curves in (a) are proportional to energy fluxes. Integrated over all angles, and averaged over time and over the globe, solar and terrestrial fluxes must balance; for this reason the two curves in (a) are drawn with equal areas. Conditions are appropriate to middle latitudes, with a solar elevation of about 40°, or for diffuse radiation.
The run-away-greenhouse effect illustrated for the terrestrial planets:

(from Atmospheres, by Goody and Walker, 1972)
Brightness temperature $T_b$:

Defining the observed flux at a wavelength $\lambda$, emitted by a planet with radius $R$ at a distance $r$, by $F_\lambda^0$, and assuming the planet emits radiation like a black-body, one can derive its \textit{brightness temperature} $T_b$ from:

$$F_\lambda^0 \equiv B_\lambda(T_b)\Omega = B_\lambda(T_b)\pi R^2/r^2$$  \hspace{1cm} (31)

with $\Omega$ the solid angle of the observed planet.

For a real planet, $T_b$ generally depends on the wavelength as well as on the phase angle.
Three regions on earth:

- Sahara
- Mediterranean
- Antarctic

(Nimbus-4 observations)
Fig. 4.21 Spectra in the thermal infrared, plotted as brightness temperatures, for four planets and Titan. Features that show as “absorptions” are formed in a region of negative temperature gradient (troposphere); those that show as “emissions” are from a warm stratosphere. [After Hanel (1983).]