Orbital Resonances

Two satellites orbiting a planet.

\[ n_1 = \frac{2\pi}{P_1} = \left( \frac{GM}{a_1^3} \right)^{\frac{1}{2}} \]

\[ n_2 = \frac{2\pi}{P_2} = \left( \frac{GM}{a_2^3} \right)^{\frac{1}{2}} < n_1 \]
A. Resonance Condition

Consider motion of #1 relative to #2:

\[ n_{rel} = n_1 - n_2 \]

or

\[ \frac{1}{P_{rel}} = \frac{1}{P_1} - \frac{1}{P_2} \]

→ satellites in conjunction at interval \( P_{rel} = \frac{2\pi}{n_{rel}} \)

What happens if \( P_{rel} = mP_1 \) with \( m \) an integer?

→ conjunctions occur exactly every \( m \) orbits of #1, at the same place in orbit #1

→ perturbations are the same every conjunction

→ effect of perturbations can grow large

\[ P_{rel} = mP_1 \quad n_{rel} = \frac{2\pi}{mP_1} = \frac{n_1}{m} \]

\[ \therefore \quad m(n_1 - n_2) = n_1 \]

\[ \therefore \quad (m-1)n_1 = mn_2 \]

i.e. \( n_1 = \frac{m}{m-1}n_2 \)

This is referred to as an m: m - 1 resonance

<table>
<thead>
<tr>
<th>m</th>
<th>resonance</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2:1</td>
<td>Ex. Io &amp; Europa</td>
</tr>
<tr>
<td>3</td>
<td>3:2</td>
<td>Ex. PSR 1257+12, Pluto</td>
</tr>
<tr>
<td>4</td>
<td>4:3</td>
<td>Ex. Titan &amp; Hyperion</td>
</tr>
</tbody>
</table>
B. Resonance Geometry

- Examine motion of Sat #1 relative to #2
- Assume initial conjunction occurs when #1 is at periapse of its elliptical orbit:

Assume \( m = 3 \)

\[
\Rightarrow P_{\text{rel}} = 3 P_1
\]

\( \Rightarrow \) conjunctions occur every 3rd periapse of #1

Also, \( (m-1)n_1 = mn_2 \)

\( \therefore (m-1)(n_1 - n_2) = n_2 \)

or \( (m-1)n_{\text{rel}} = n_2 \)

i.e. \( n_{\text{rel}} = \frac{n_2}{m-1} \)

or \( P_{\text{rel}} = (m-1)P_2 \)

\( \Rightarrow \) conjunctions also occur every 2 orbits of #2.
C. Resonance Effects & Examples

1. Resonances provide stable orbits in which planets, satellites, or asteroids may be trapped, and protected against close encounters or large perturbations to the orbit. Examples:

<table>
<thead>
<tr>
<th>2:1</th>
<th>Io – Europa</th>
<th>2:1 resonances</th>
</tr>
</thead>
<tbody>
<tr>
<td>2:1</td>
<td>Europa - Ganymede</td>
<td></td>
</tr>
<tr>
<td>3:2</td>
<td>Mimas - Tethys</td>
<td>3:2 resonances</td>
</tr>
<tr>
<td>3:2</td>
<td>Enceladus - Dione</td>
<td></td>
</tr>
<tr>
<td>*</td>
<td>*Griqua - Jupiter</td>
<td></td>
</tr>
<tr>
<td>*</td>
<td>*Hilda - Jupiter</td>
<td></td>
</tr>
<tr>
<td>4:3</td>
<td>Neptune - Pluto</td>
<td>4:3 resonances</td>
</tr>
<tr>
<td>4:3</td>
<td>Titan - Hyperion</td>
<td></td>
</tr>
<tr>
<td>*</td>
<td>*Thule - Jupiter</td>
<td></td>
</tr>
</tbody>
</table>

2. Resonances can increase orbital eccentricity or inclination, which may lead to enhanced tidal energy dissipation. (See TIDES notes).

Examples:
- Io: forced $e \Rightarrow$ volcanism
- Miranda: melted interior?
- Europa: melted surface/$\text{H}_2\text{O}$ “lava”?
- Enceladus: South polar plumes
D. Satellite Resonances

Consider the perturbation of an inner satellite on an eccentric orbit by a more massive outer satellite \( m_2 \) (e.g., Enceladus & Dione). The disturbing function for body 1 is

\[
\mathbf{R}_1 = Gm_2 \left( \frac{1}{\Delta} - \frac{r_1 \cos \phi}{r_2^2} \right)
\]

For zero inclinations, \( \Delta = \left( r_1^2 + r_2^2 - 2r_1 r_2 \cos \phi \right)^{1/2} \)

and \( r_1 = \frac{a_1(1-e_1^2)}{1+e_1 \cos \nu_1}, \quad r_2 = \frac{a_2(1-e_2^2)}{1+e_2 \cos \nu_2}, \quad \phi = \nu_2 - \nu_1 \)

The indirect term arises from the displacement of \( M \) away from the center of mass due to the presence of \( m_2 \). In general, the expansion of \( \mathbf{R}_1 \) in terms of orbital elements (via \( r_1, r_2, \nu_1 \) and \( \nu_2 \)) is messy.

It may be written as a Fourier-like series in the mean longitudes \( \lambda, \varpi, \) and \( \Omega \):

\[
\mathbf{R}_1 = \frac{Gm_2}{a_2} \sum \left( \frac{a_1}{a_2} \right)^l F_{lm} (i_1) F_{lm'} (i_2) G_{lpq} (e_1) G_{lp'q'} (e_2) \cos \Psi
\]

where \( \Psi_{lmpp'qq'} = (l-2p+q)\lambda_1 - (l-2p'+q')\lambda_2 - q\varpi_1 + q'\varpi_2 - (l-2p-m)\Omega_1 + (l-2p'-m)\Omega_2 \)

and the sum is over \( 2 \leq l < \infty, \quad 0 \leq m \leq l, \quad 0 \leq p \leq l, \quad 0 \leq p' \leq l, \quad -\infty \leq q \leq \infty, \quad -\infty \leq q' \leq \infty. \)
Some simplification arises for small $e$ and/or $i$,
as $F_{imp}(i) \propto (\sin i)^{|l-2p-m|}$
and $G_{eq}(e) \propto e^{q}$
i.e., the leading powers of $\sin i$ and $e$ are equal (in absolute value) to the coefficients of $\Omega$ and $\sigma$ in $\psi$.
Note that the sum of all the coefficients in $\psi$ is zero (known as d'Alembert's rule), which arises from rotational symmetry in the problem.
The sum of the coefficients of $\Omega$ and $\Omega'$ is even (due to North - South symmetry).
From the above, and for small $e$ and $i$, the dominant terms in $\mathbb{R}_1$ have arguments

$$\psi=(m+q)\lambda_1-(m+q')\lambda_2-q\sigma_1-q'\sigma_2$$

with $q$ and $q'$ equal to 0 or $\pm 1$, or more explicitly:

$$\begin{cases} 
\psi_1 = m(\lambda_1 - \lambda_2) & \cdots \mathbb{R}_1 \text{ is independent of } e_1 \text{ and } e_2 \\
\psi_2 = (m \pm 1)\lambda_1 - m\lambda_2 \mp \sigma_1 & \cdots \mathbb{R}_1 \propto e_1 \\
\psi_3 = m\lambda_1 - (m \pm 1)\lambda_2 \pm \sigma_2 & \cdots \mathbb{R}_1 \propto e_2
\end{cases}$$
In general, all of these perturbing terms affect the motion of $m_1$, leading to a complex super-position of perturbations at different frequencies. Perturbations can be especially large if the frequency $\psi$ is unusually small, as the perturbing forces act in the same directions for a long term. This is what is meant by an orbit-orbit resonance.

Since $\dot{\lambda}_1 \approx n_1$ and $\dot{\lambda}_2 \approx n_2 < n_1$ we see that the above arguments can be resonant only if:

$$\psi_1 \approx 0 \Rightarrow n_1 \approx n_2 \Rightarrow a_1 \approx a_2$$

$$\psi_2 \approx 0 \Rightarrow n_1 \approx \frac{mn_2 - \dot{\phi}_1}{m-1} \approx \left( \frac{m}{m-1} \right) n_2$$

$$\psi_3 \approx 0 \Rightarrow n_1 \approx \frac{(m+1)n_2 - \dot{\phi}_2}{m} \approx \left( \frac{m+1}{m} \right) n_2$$

The 1st case can lead to resonance only for co-orbital satellites or asteroids (e.g., Trojans, or Janus & Epimetheus).

The 2nd and 3rd cases are referred to as first order resonances, since the coefficients of $\lambda_1$ and $\lambda_2$ differ by 1 and account for many of the observed asteroid, satellite, and ring resonances. We will look further at the 2nd case, for which

$$R_1 = -\frac{Gm_2}{a_1} e_1 \alpha F_m(\alpha) \cos \psi_2,$$

where $\alpha \equiv a_1 / a_2$. 
For a first-order ILR, $\Omega_p = n_s$ and the relevant averaged term from the disturbing function is given by

$$\langle R \rangle_e = \frac{G m_s}{a_s^2} f(\alpha) e \cos \varphi,$$

where the subscript $s$ denotes the perturbing satellite, $\alpha = a/a_s$, $f(\alpha) = \frac{1}{2}(-2m - \alpha D)k_{1/2}^{(m)}$, and $\varphi$ is the resonant argument, given by

$$\varphi = (m - 1)\lambda - m\lambda_s + \bar{\varphi}.$$  \hspace{1cm} (17)

Here $m$ is a positive integer. The 'exact resonance' is defined by the unperturbed expression

$$\dot{\varphi} = (m - 1)n - mn_s + \bar{\varphi}_{sec} = 0,$$

where $\bar{\varphi}_{sec}$ is the secular pericenter precession rate, generally due to $J_2$ and higher zonal gravity harmonics.

To zeroth order in inclination and lowest order in eccentricity, Lagrange's equations for this perturbation are given by

\begin{align*}
\dot{n} &= -3(m - 1)\beta n^2 e \sin \varphi \hspace{1cm} (19) \\
\dot{e} &= -\beta n \sin \varphi \hspace{1cm} (20) \\
\dot{\omega} &= -\frac{\beta n}{e} \cos \varphi + \bar{\omega}_{sec} \hspace{1cm} (21) \\
\dot{e} &= -\frac{1}{2} \beta n e \cos \varphi \hspace{1cm} (22)
\end{align*}

where $\beta = -(m_s/M)\alpha f(\alpha)$ is the dimensionless resonance strength and $M$ refers to the central mass.

Kepler's third law has been used in its usual form $(GM = n^2a^3)$ to eliminate $G$, and in its differential form $(3n\dot{a} = -2n\dot{a})$ to rewrite the first equation in terms of orbital mean motion $n$. The general expression for $\dot{\varphi}$ is then

$$\dot{\varphi} = (m - 1)n - mn_s + \bar{\varphi}_{sec} - \beta n \left[ \frac{1}{e} + \frac{(m - 1)e}{2} \right] \cos \varphi.$$  \hspace{1cm} (23)

Dividing equations 19 and 20 gives

$$\frac{dn}{de} = 3(m - 1)ne,$$

\hspace{1cm} (24)
which, when \( n(e = 0) = n_0 \), has the exact solution

\[
n = n_0 \exp \left[ \frac{3}{2} (m - 1) e^2 \right]
\]  

(25)

and the approximate solution

\[
n \approx n_0 \left[ 1 + \frac{3}{2} (m - 1) e^2 \right].
\]  

(26)

Equation 23 then becomes

\[
\dot{\phi} = (m - 1)n_0 \left[ 1 + \frac{3}{2} (m - 1) e^2 \right] - mn_s + \omega_{\text{sec}} - \beta n_0 \left[ 1 + \frac{3}{2} (m - 1) e^2 \right] \left[ \frac{1}{e} + \frac{(m - 1) e}{2} \right] \cos \phi.
\]  

(27)

To lowest (non-vanishing) order in eccentricity this is

\[
\dot{\phi} = (m - 1)n_0 \left[ 1 + \frac{3}{2} (m - 1) e^2 \right] - mn_s + \omega_{\text{sec}} - \frac{\beta n_0}{e} \cos \phi.
\]  

(28)

Note that for the terms with \( \beta \), the lowest-order term in \( e \) was \( \propto 1/e \). Dropping the other terms is justified because \( \beta \) is a very small quantity and thus \( O(\beta n_0 e) \ll O(n_0 e^2) \). Defining a new parameter as

\[
\nu \equiv (m - 1)n_0 - mn_s + \omega_{\text{sec}},
\]  

(29)

equation 28 becomes

\[
\dot{\phi} = \nu + \frac{3}{2} (m - 1)^2 n_0 e^2 - \frac{\beta n_0}{e} \cos \phi.
\]  

(30)

This new angular frequency \( \nu \) is therefore a measure of distance from the 'exact' resonance. From the definition of exact resonance in equation 18, we obtain

\[
\nu = (m - 1)(n_0 - n_{\text{res}}).
\]  

(31)

From Kepler's third law, it is clear that \( \nu < 0 \) is exterior to the resonance, and \( \nu > 0 \) is interior to the resonance. This can be made even more transparent by rewriting \( \nu \) in terms of semi-major axis rather than mean motion,

\[
\nu = (m - 1)n_{\text{res}} \left[ \frac{n_0}{n_{\text{res}}} - 1 \right]
\]  

(32)
\begin{align*}
&= (m - 1)n_{\text{res}} \left[ \left( \frac{a_0}{a_{\text{res}}} \right)^{-3/2} - 1 \right] \\
&= (m - 1)n_{\text{res}} \left[ 1 + \frac{a_0 - a_{\text{res}}}{a_{\text{res}}} \right]^{-3/2} - 1 \\
&\approx \frac{3n_{\text{res}}}{2a_{\text{res}}} (m - 1)(a_0 - a_{\text{res}}) .
\end{align*}

2.3 Equilibrium Solutions

Equation 30 has equilibrium solutions when

\[ \nu = \frac{\beta n_0}{\epsilon} \cos \varphi - \frac{3}{2} (m - 1)^2 n_0 \epsilon^2 . \]

The \( \cos \varphi \) can be dealt with by considering Eqn. 20, which only has equilibrium solutions for \( \varphi = 0 \) or \( \varphi = \pi \). Therefore \( \cos \varphi = \pm 1 \) and can be absorbed into \( \epsilon \) in equation 36, where \( \epsilon > 0 \) means \( \varphi = 0 \) and \( \epsilon < 0 \) means \( \varphi = \pi \). The condition for equilibrium then assumes the form of a simple cubic equation,

\[ \frac{3}{2} (m - 1)^2 \epsilon^3 + \beta \epsilon - \beta = 0 , \]

where \( \bar{\nu} \equiv \nu/n_0 \). The roots of this equation therefore correspond to equilibrium points, and the number of real-valued roots will vary with the parameter \( \bar{\nu} \). (Recall that decreasing \( \bar{\nu} \) has the meaning of moving outward in semi-major axis; cf. Eqn. 35.) Figure 3 plots the left-hand side of Eqn. 37 for several values of \( \bar{\nu} \), using \( m = 2 \) and \( \beta = 4.95 \times 10^{-8} \) (the appropriate values for Mimas’s perturbations at the B ring edge).

For large positive \( \bar{\nu} \), there is a single equilibrium point where \( \epsilon \) is small and \( \varphi = 0 \). As \( \bar{\nu} \) decreases, this equilibrium point always exists as it moves to ever larger \( \epsilon \). When \( \bar{\nu} \) becomes less than a critical value \( \bar{\nu}_c \), two new equilibrium points appear where \( \varphi = \pi \). As \( \bar{\nu} \) continues to decrease, one of the new equilibrium points moves to ever smaller \( |\epsilon| \), while the other moves to ever larger \( |\epsilon| \). The critical point itself occurs at

\begin{align*}
\bar{\nu}_c &= -\frac{1}{2} (9(m - 1)\beta)^{2/3} \\
\epsilon_c &= -\left( \frac{\beta}{3(m - 1)^2} \right)^{1/3} .
\end{align*}

An alternative method of investigating Eqn. 37 is to plot the equilibrium solutions as a bifurcation diagram of \( \epsilon \) versus \( \bar{\nu} \); this is done in Figure 3. The upper branch corresponds to the equilibrium point where \( \varphi = 0 \), while the forking lower branch corresponds to the equilibrium points where \( \varphi = \pi \). The
Figure 2: Equilibrium points for the Mimas 2:1 ILR correspond to zero-crossings of $F'(e)$. The dot-dashed line is for $\tilde{\nu} = +10^{-4}$, the solid line is for $\tilde{\nu} = 0$, the dashed line is for $\tilde{\nu} = \tilde{\nu}_c$, and the dotted line is for $\tilde{\nu} = -10^{-4}$. Note the bifurcation that occurs when two new equilibrium points appear as $\tilde{\nu}$ drops below $\tilde{\nu}_c$. The symbols marking the equilibrium points correspond to those in Figure 3. The values of $\tilde{\nu}$ plotted are those used as examples in Figures 4–11.
lower branch appears in a saddle-node bifurcation as $\bar{\nu}$ decreases past the critical value. The portion of this branch where $|e| > |e_c|$ can be shown to be unstable. For a given value of $\bar{\nu}$, this diagram tells us what the equilibrium eccentricities are. That is, given the location (semi-major axis) of a test particle relative to the resonance, we know what the forced eccentricities due to the resonance are.

![Graph showing bifurcation diagram for the Mimas 2:1 ILR.](image)

Figure 3: Bifurcation diagram for the Mimas 2:1 ILR. The solid line is the exact solution of Eqn. 37, the dotted line is the asymptotic solution for small $e$ (Eqn. 40), and the dashed line is the asymptotic solution for large $e$ (Eqn. 41). The lower branch ($\varphi = \pi$) of the exact solution is unstable when $|e| > |e_c|$. Asterisks indicate the bifurcation, or critical, point (see text).

We can also derive asymptotic forms of Eqn. 37 when $e$ is small or large, roughly corresponding to its size relative to $e_c \sim \beta^{1/3}$. For $e \ll e_c$, the cubic leading term can be neglected, giving

$$ e \sim \frac{\beta}{\bar{\nu}}. \quad (40) $$

For $e \gg e_c$, the constant $\beta$ term can be neglected, giving

$$ e \sim \pm \frac{1}{m-1} \left( -\frac{2}{3} \bar{\nu} \right)^{1/2}. \quad (41) $$

Equations 40 and 41 are also plotted in Figure 3. Looking back at Eqns. 19 and 21, we see that for the
small $e$ case, the resonant perturbation on $\varpi$ is dominant over that on $n$. Similarly, we see that in the large $e$ case, the resonant perturbation on $n$ is dominant over that on $\varpi$.

Neither asymptotic form is appropriate when $\tilde{\nu} = 0$, but we see directly from Eqn. 37 that $\varphi_0 = 0$ and

$$ e_0 = \left( \frac{2\beta}{3(m-1)^2} \right)^{1/3}. \quad (42) $$

### 2.4 Oscillations Around Equilibrium

We can now examine the motion for small displacements from the equilibrium points, again applying Eqns. 30 and 20 (for $m = 2$) to get

$$ \dot{\varphi} = \nu + \frac{3}{2} n_0 e^2 - \frac{\beta n_0}{e} \cos \varphi \quad (43) $$

$$ \dot{e} = -\beta n_0 \sin \varphi, \quad (44) $$

where the replacement of $n$ by $n_0$ in the second equation is again justified by the smallness of $\beta$.

**Small $e$**

For the small $e$ case, these equations become

$$ \dot{\varphi} \approx \nu - \frac{\beta n_0}{e} \cos \varphi \quad (45) $$

$$ \dot{e} = -\beta n_0 \sin \varphi. \quad (46) $$

The solutions to these equations become more transparent by transforming to the new variables

$$ h = e \cos \varphi \quad (47) $$

$$ k = e \sin \varphi, \quad (48) $$

which lead to

$$ \dot{h} = -\nu k \quad (49) $$

$$ \dot{k} = \nu h - \beta n_0. \quad (50) $$
Recalling that the equilibrium solution for \( \nu > 0 \) is \( e_0 = \beta n_0 / \nu \) with \( \varphi = 0 \), we choose initial conditions of \( e = e_0 + e_f \) and \( \varphi = 0 \), where \( e_0 \) is the equilibrium (forced) eccentricity and \( e_f \) is the free eccentricity. This results in the solutions

\[
\begin{align*}
h & = e_0 + e_f \cos \nu t \\
\dot{k} & = e_f \sin \nu t.
\end{align*}
\] (51)

This describes anticlockwise circular motion of radius \( e_f \) and angular speed \( \nu \) about the point \( e_0 \) on the positive \( h \)-axis. Recalling that \( e \) and \( \varphi \) are the equivalent of polar coordinates on the \( hk \)-plane, we see that \( e \) and \( \varphi \) oscillate around the equilibrium point with frequency \( \nu \). As long as \( e_f < e_0 \), this motion describes libration of \( \varphi \). When \( e_f > e_0 \), the circle includes the origin, and \( \varphi \) instead circulates. Note that the above analysis also holds for \( \nu < 0 \) with initial condition \( \varphi = \pi \), describing clockwise circular motion about the point \( e_0 \) on the negative \( h \)-axis.

**Large \( e \)**

For the large \( e \) case, the equations become

\[
\begin{align*}
\dot{\varphi} & \approx \nu + \frac{3}{2} n_0 e^2 \\
\dot{e} & = -\beta n_0 \sin \varphi.
\end{align*}
\] (53)

Recalling the equilibrium solution is \( e_0 = (-2\nu/3n_0)^{1/2} \), the first equation can be rewritten as

\[
\dot{\varphi} = \nu \left[ 1 - \left( \frac{e}{e_0} \right)^2 \right].
\] (55)

If we assume solutions of the form

\[
\begin{align*}
\varphi & = \varphi_f \sin \omega t \\
e & = e_0 + e_f \cos \omega t,
\end{align*}
\] (56)

substituting into the equations of motion yields

\[
\omega = \left( -6\beta^2 n_0^3 \nu \right)^{1/4}
\] (58)
\[ \frac{e_f}{\varphi_f} = \left( -\beta^2 n_0 / 6 \nu \right)^{1/4}, \] (59)

provided that \( e_f \ll e_0 \) and \( \varphi_f \ll 1 \). The first condition is reasonable because the equilibrium value is assumed large to start. Compared to the small-\( e \) libration frequency \( \nu \), \( \omega \) is both smaller and increases more slowly with increasing distance from the resonance (\( \propto \nu^{1/4} \)).

### 2.5 Numerical Illustration

To provide concrete examples of motion near a first-order ILR, we examine the obvious case of the 2:1 resonance with Mimas. To start simply, Mimas is placed on a circular orbit and the oblateness of Saturn is suppressed. Mimas (\( m_s/M = 6.5969 \times 10^{-8} \)) then has a semi-major axis \( a_s = 185,520 \) km and mean motion \( n_s = 381.5475^\circ \) day\(^{-1} \), leading to the 'exact' resonance values of \( a_{\text{res}} = 116,870.28 \) km and \( n_{\text{res}} = 763.0950^\circ \) day\(^{-1} \) when \( m = 2 \). Then \( \alpha = 0.6300 \), \( f(\alpha) = -1.1907 \), and \( \beta = 4.95 \times 10^{-8} \). Also, \( \nu = -0.00979 (a_0 - a_{\text{res}}) \) day\(^{-1} \), where \( a \) is in kilometers, so that \( \bar{\nu} = -1.283 \times 10^{-5} (a_0 - a_{\text{res}}) \) day\(^{-1} \).
Figure 1: Solid line is the exact solution of equation 22. Broken lines are asymptotic solutions (equations 23 and 24). Solutions for \( e > e_e \) and \( \varphi = \pi \) are unstable.

Figure 2: Dotted lines are for \( \bar{\nu} = \pm 10^{-4} \), dashed line is for \( \bar{\nu} = \bar{\nu}_e \), and solid line is for \( \bar{\nu} = 0 \). Equilibrium points correspond to zero-crossings of \( F(e) \). Symbols correspond to equilibrium points in figure 1.
Figure 3: $h$-$k$ plot for $\bar{\nu} = +10^{-4}$. Motion is anticlockwise.

Figure 4: $h$-$k$ plot for $\bar{\nu} = 0$. Motion is anticlockwise.
Figure 10: $h$-$k$ plot for $\bar{\nu} = -10^{-4}$. Motion is clockwise around the small-$e$ equilibrium point, anticlockwise otherwise.
**Solar System Resonances**

Important orbital resonances in the solar system include: (all listed arguments librate)

<table>
<thead>
<tr>
<th>Category</th>
<th>Equation</th>
<th>Phase</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Planets</strong></td>
<td>$2\lambda_n - 3\lambda_p + \varpi_p$</td>
<td>Trojans</td>
<td>0th order</td>
</tr>
<tr>
<td><strong>Asteroids</strong></td>
<td>$\lambda - \lambda_j$</td>
<td>...</td>
<td>1st order</td>
</tr>
<tr>
<td></td>
<td>$3\lambda - 4\lambda_j + \varpi$</td>
<td>Thule</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$2\lambda - 3\lambda_j + \varpi$</td>
<td>Hilda</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda - 2\lambda_j + \varpi$</td>
<td>Griqua</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda - 3\lambda_j + 2\varpi$</td>
<td>Alinda</td>
<td>2nd order</td>
</tr>
<tr>
<td><strong>Jovian Satellites</strong></td>
<td>$\lambda_I - 2\lambda_E + \varpi_I$</td>
<td></td>
<td>1st order</td>
</tr>
<tr>
<td></td>
<td>$\lambda_I - 2\lambda_E + \varpi_E$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_E - 2\lambda_G + \varpi_E$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lambda_I - 3\lambda_E + 2\lambda_G$</td>
<td></td>
<td>0th order (The LAPLACE RELATION)*</td>
</tr>
<tr>
<td><strong>Saturnian satellites</strong></td>
<td>$2\lambda_m - 4\lambda_T + \Omega_M + \Omega_T$</td>
<td></td>
<td>2nd order</td>
</tr>
<tr>
<td></td>
<td>$\lambda_E - 2\lambda_D + \varpi_E$</td>
<td></td>
<td>1st order</td>
</tr>
<tr>
<td></td>
<td>$3\lambda_{Ti} - 4\lambda_H + \varpi_H$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*This relation follows from the two above, and is an example of a “3-body resonance”. It is equal to 180°, and prevents all three satellites ever being in conjunction simultaneously.*
In addition, there are several important near-resonant arguments which circulate slowly:

<table>
<thead>
<tr>
<th>Planets</th>
<th>(2\lambda_J - 5\lambda_s)</th>
<th>The Great Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\lambda_U - 2\lambda_N)</td>
<td></td>
</tr>
<tr>
<td><strong>Uranian satellites</strong></td>
<td>(\lambda_A - 2\lambda_U)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2\lambda_T - 3\lambda_0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\lambda_M - 3\lambda_A + 2\lambda_U)</td>
<td>cf. Laplace relation</td>
</tr>
</tbody>
</table>

The so-called “Nice model” of the early solar system presumes that at one time Jupiter and Saturn passed through a 2:1 resonance resulting in large perturbations which destabilized many asteroids and “ejected” Uranus and Neptune to their current orbits.
Saturn's e

GREAT INEQUALITY

900-year period
Horseshoe Trajectory of 2000PH5 in a Frame Co-rotating with Earth.

@ at center, $\bullet$ at $X=1$.

$0.994 < a < 1.006$ AU

$e = 0.230, \ i = 183^\circ$

~100 year trajectory with 2005 in Red.

* will leave horseshoe orbit in ~100 years after close encounter ("bounce") with Earth, it will then be in a passing orbit.
FIG. 1. Nicholson et al.: "Saturn's coorbital satellites"
Kirkwood gaps in Asteroid Belt
Neptune Resonances

![Graph showing Neptune resonances with labeled axes and markers for different classifications such as SDO, Detached, Resonant, and Classical.](image-url)
Major resonances in Saturn’s rings