Problem 1. In class we showed that the expressions for seismic velocities $V_p$ and $V_s$ can be combined with the equation of hydrostatic equilibrium, $dP/dr = -\rho g$, to obtain

$$\frac{d\rho}{dr} = \frac{-g(r)\rho(r)}{\Phi(r)},$$

where $\Phi = V_p^2 - \frac{4}{3} V_s^2$. What two assumptions are implicit in this equation?

(a) As a rough approximation to the real seismic velocity profiles, we may write for the Earth’s mantle

$$V_p \approx 8100 + 0.002z \text{ m/s},$$
$$V_s \approx 4500 + 0.001z \text{ m/s},$$

where $z = R_\oplus - r$ is the depth in meters below the Earth’s surface. Write a computer program to calculate $\rho(r)$, $g(r)$, $P(r)$ and $m(r)$ through the mantle, taking as starting conditions at $r_1 = R_\oplus = 6371$ km the values $m_1 = M_\oplus = 5.9736 \times 10^{27}$ g, $P_1 = 0$, and $\rho_1 = 3.35$ g/cm$^3$ (the density of the upper mantle).

(b) Evaluate $\rho_c$, $g_c$, and $P_c$ at the core-mantle boundary, $r_c = 3480$ km. What fraction of the Earth’s mass remains in the core, according to your model?

(c) Inside the core we may write approximately

$$V_p \approx 11,500 - 0.001r \text{ m/s}, \quad V_s \approx 0.0,$$

where the radius $r$ is expressed in meters and we will ignore the solid inner core for simplicity. At the core-mantle boundary the density abruptly changes by an unknown amount $\Delta \rho$. Starting at $r = 0$ with $m(r) = 0$ and an assumed central density $\rho_0$, integrate outwards to $r = r_c$ to obtain $\rho(r)$, $m(r)$, and $g(r)$ for the core. Adjust $\rho_0$ and repeat the calculation until the mass of the core, $m(r_c)$ matches that obtained in part (b), to an accuracy of at least 1%. What value of $\rho_0$ is required and what is the density contrast at the core/mantle boundary, $\Delta \rho$? (This is known as the “shooting technique” for solving a differential equation with boundary conditions at both ends.)

(d) Plot $\rho(r)$, $P(r)$, $m(r)/M_\oplus$ and $g(r)$ for $0 < r < R_\oplus$. What is the pressure at the center of your model Earth, $P_0$? (Note that pressures are additive and continuous, so you can integrate the hydrostatic equation outwards through the core, starting with any convenient value of $P_0$ — even zero — and then add a constant in order to match $P_c$ from part (b).)

(e) Calculate the moment of inertia factor, $C/MR^2$ for your model Earth, and compare it with the observed value of 0.331. Is your model in reasonable agreement with observations? Recall that for a sphere

$$A = B = C = \frac{2}{3} \int_0^R r^2 \, dm = \frac{8\pi}{3} \int_0^R r^4 \rho(r) \, dr.$$
**Problem 2.** We look at a simple model for the equilibrium thermal gradient inside a solid planet due to radioactive heat sources. Consider a spherical planet of radius \( R \) which loses heat from its interior by conduction. The heat is generated by the decay of radioactive isotopes, whose local power output per unit mass is denoted by \( Q \), in units of W/kg (*not to be confused with the planet’s tidal \( Q \)!).

(a) Following the nomenclature used for stars, we denote the total rate at which energy is transported outwards by the luminosity \( L \), expressed in Watts. In terms of the flux of energy \( F \) at radius \( r \) we have \( L(r) = 4\pi r^2 F(r) \), where \( F \) is in the usual units of W/m\(^2\). If the planet is neither warming up nor cooling, show that
\[
\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)Q(r),
\]
where \( \rho(r) \) is the interior density profile. For conductive heat transport, we can write \( F = -K(dT/dr) \), where \( K \) is the thermal conductivity. Show that the temperature profile is then governed by the differential equation:
\[
\frac{d^2T}{dr^2} + 2 \frac{dT}{dr} = -\frac{Q\rho}{K}.
\]
We will look at two simple analytic solutions of this equation.

(b) First consider the case \( Q(r) = 0 \), and show that the most general solution is of the form
\[
T(r) = \frac{A}{r} + B,
\]
where \( A \) and \( B \) are arbitrary constants. Clearly the second term corresponds to an isothermal planet with no internal heat sources. Can you suggest a physical — albeit unrealistic — interpretation of the first term? (Hint: evaluate the luminosity, \( L(r) \) corresponding to this solution.)

(c) Now consider the more interesting case \( Q(r)\rho(r) = \gamma \), where \( \gamma \) is a constant. Look for a solution of the form \( T(r) = \alpha r^\beta \), and show that
\[
T(r) = -\frac{\gamma}{6K}r^2
\]
satisfies the D.E. The most general solution is of course a combination of this particular solution plus the homogeneous solution, \( A/r + B \), derived above.

(d) Sketch a plot of your solution, subject to the boundary conditions that \( L \to 0 \) as \( r \to 0 \) and \( T(R) = T_s \) at the surface of the planet. Evaluate the constants \( A \) and \( B \) for this case, and show that the central temperature of the planet is given by
\[
T_0 \equiv T(0) = T_s + \frac{\gamma R^2}{6K}.
\]
Verify that the interior luminosity for this model is
\[ L(r) = \frac{4\pi r^3}{3} \gamma, \]
as expected for a uniformly-heated sphere.

(e) Evaluate \( T_0 \) for the Moon, assuming that \( \rho = 3500 \text{ kg/m}^3 \), \( Q = 5 \times 10^{-12} \text{ W/kg} \) (an average value for meteorites), \( K = 3.0 \text{ W/m/Kelvin} \) (typical of granite) and \( T_s = 250 \text{ K} \).

(f) Repeat your calculation for Pluto, assuming a 50/50 ice-rock mixture with \( \rho = 2000 \text{ kg/m}^3 \), \( Q = 2.5 \times 10^{-12} \text{ W/kg} \), \( K = 2.0 \text{ W/m/Kelvin} \) and a surface temperature \( T_s = 40 \text{ K} \).

**Problem 3.** The moment of inertia of a planet, \( C \), in concert with its mean density \( \rho \), impose a strong constraint on its interior density distribution. In this problem we examine one of the simplest possible planetary models: a spherical planet with a central core of radius \( r_c \) and uniform density \( \rho_c \), surrounded by a shell (the ‘mantle’) of density \( \rho_m \) and exterior radius \( R \). The model thus has three free parameters, which may be adjusted (but not uniquely) to fit the observed values of \( \rho \) and \( \alpha = C/MR^2 \) (note that the total mass, \( M = (4/3)\pi \rho R^3 \) is determined by \( R \) and \( \rho \) and is not an an independent observable).

(a) Derive analytic expressions for \( \rho \) and \( \alpha \) in terms of \( \rho_c \), \( \rho_m \) and the fractional core radius, \( x = r_c/R \). Use these to show that \( x \) and \( \rho_c \) can be expressed in terms of the single parameter \( \rho_m \) and the observables \( \rho \) and \( \alpha \):

\[ x = \left( \frac{5}{2} \alpha \rho - \rho_m \right)^{\frac{1}{2}} \left( \rho - \rho_m \right) \]

and

\[ \Delta \rho \equiv \rho_c - \rho_m = \frac{(\rho - \rho_m)^{\frac{5}{2}}}{(\frac{5}{2} \alpha \rho - \rho_m)^{\frac{3}{2}}}. \]

This can be thought of as a 1-parameter family of models with varying mantle density.

(b) Show that there are two limiting cases of this family of models: one corresponding to \( \rho_m \rightarrow 0 \), and the other to \( \rho_m \rightarrow 5\alpha \rho/2 \). What are the values of \( x \) and \( \rho_c \) for each of these cases. Can you describe these limits in sensible physical terms?

(c) Calculate the mass of the core \( M_c \) for the 2-layer model, and show that it can be expressed in the form

\[ M_c = \frac{4\pi}{3} R^3 \left[ \rho - \rho_m (1 - x^3) \right]. \]

What are the values of \( M_c \) for each of the limiting cases in part (c)? Do your answers make physical sense?

(d) Calculate and plot \( x \) and \( \rho_c \) vs \( \rho_m \) for a planet with \( \rho = 0.69 \) and \( \alpha = 0.24 \) (the Saturn “family”).
Problem 4. A notable feature of the Earth’s gravity field, and to a lesser extent that of Venus and Mars, is that there is very little obvious correlation between the local value of \( g \), the gravitational acceleration, and topography. This phenomenon is known as ‘isostacy’, and seems to imply approximately the same mass per unit surface area, independent of the local elevation. We will explore a very simple isostatic model referred to as the Airy model, after the 19C British astronomer and geophysicist. Airy postulated that, at a given radius within the Earth, somewhere well below the crust and known as the compensation depth, the total mass of a column of material above that level per unit area is the same everywhere. (This is basically Archimedes’ principle and amounts to assuming that the crust ‘floats’ in the mantle, the way an iceberg floats in the ocean.)

(a) Consider first the balance between continents and ocean basins. Assume that the average density of the continental crust (mostly granite) is \( \rho_g = 2690 \text{ kg/m}^3 \), while that of oceanic crust (mostly basalt) is \( \rho_b = 3010 \text{ kg/m}^3 \). The density of seawater is \( \rho_w = 1028 \text{ kg/m}^3 \) (all numbers courtesy of Wikipedia.) Assume further that the average thickness of the oceanic crust is \( t_{oc} = 10 \text{ km} \), that the average depth of the oceans is \( d = 4 \text{ km} \), and that the average altitude of the continents is near sea-level. Use Airy’s principle of isostacy to show that the average thickness of the continental crust, \( t_{cc} \) is given by

\[
t_{cc} = \frac{[(\rho_m - \rho_w)d + (\rho_m - \rho_b)t_{oc}]/(\rho_m - \rho_g)}{\rho_m - \rho_g}.
\]

Use the above numbers to estimate \( t_{cc} \) in km. You may assume a density for the Earth’s upper mantle of \( \rho_m = 3300 \text{ kg/m}^3 \). (It does not matter exactly where you place the compensation depth, as long as it is below both the continental and oceanic crust.)

(b) The Airy model also implies that the continental crust is even thicker under high mountain ranges such as the Himalayas or the Andes. Redo your calculation for the thickness of the crust for a region where the average elevation is \( h = 4500 \text{ m} \) above sea-level. This thickening is sometimes referred to as the ‘roots’ of a mountain range, and is readily seen in seismic data for extended mountainous regions.

(c) Use your results in parts (a) and (b) above to make a simplified sketch showing a north-south cross-section of both the surface topography and the depth of the Moho (the boundary between the crust and the mantle) from the Indian ocean across the Indian subcontinent and into the Himalayas. Make sure to label the vertical scale of your plot and to show all features at a common scale.