Droplet growth by condensation

\[ n = \frac{\#}{V} \quad N = \# \]

\[ \bar{V} = \text{flow velocity} \quad \text{here assume} \quad \bar{V} = 0 \]

1. Where does one start? General equations for safety

\[ \frac{\partial n_i}{\partial t} + \nabla \cdot (\bar{V} n_i) = \nabla \cdot (D_i \nabla n_i) + S_o - S_i \quad \text{source} \quad \text{sink} \quad \text{gas} \]

\[ \frac{\partial N}{\partial t} = + 4\pi r^2 D_i \frac{\partial n_i}{\partial r} \quad \text{droplet} \]

\[ \frac{\partial}{\partial t} (n c_p T) + \nabla \cdot (\bar{V} n c_p T) = \nabla \cdot (D_h n c_p \nabla T) \quad \text{gas heat} \]

\[ \frac{1}{\partial t} (c_p T_n N) = + 4\pi r^2 D_h \frac{\partial T}{\partial r} + L \frac{\partial N}{\partial t} \quad \text{droplet heat} \]

\[ \frac{d p_n}{d \bar{T}} = \frac{\rho a}{T R_i \bar{T}} \quad C - C \]

2. Assumptions

Stationary droplet (falling slowly) \( \Rightarrow \bar{V} = 0 \)

\( \frac{\partial n_i}{\partial t} \) negligible, Flux goes to droplet \( \Rightarrow 4\pi r^2 D_i \frac{\partial n_i}{\partial r} = \text{constant} \)

Fixed supersaturation at \( r = \infty \), and \( T = 0 \)
3. **Fluxes + constraints.**

\[
4\pi r^2 D_i \frac{\partial n_i}{\partial r} = 4\pi r_0^2 D_i \frac{n_{i,\infty} - n_{i,0}}{r_0} \quad \text{(neglect } \frac{\partial n}{\partial t})
\]

[Comment: \( \frac{\partial n_i}{\partial r} = \frac{r_0}{r^2} (n_{i,\infty} - n_{i,0}) \Rightarrow n_i = \frac{r_0}{r} (n_{i,\infty} - n_{i,0}) + n_{i,0} \)]

**Latest heat warms droplet.** \( ST = \) increment in \( T \),

**Heat balance**

\[
4\pi r_0^2 D_H n c_p \frac{ST}{r_0} = \dot{N} L \quad \text{(a)} \quad \frac{dN}{dt}
\]

\[
\frac{S n_{i,0}}{n_{i,0}} = \frac{\dot{S} n_0}{n_0} = \frac{L}{R_i T} \frac{ST}{T} \quad \text{(b)}
\]

\[
\uparrow \text{Reduces supersaturation by raising } n_{i,0}
\]

\[
\therefore \quad \frac{dN}{dt} = 4\pi r_0^2 D_i \frac{n_{i,\infty} - S n_{i,0} - n_{i,0}}{r_0} \quad \text{flat value} \quad \text{(c)}
\]

To finish, solve (a), (b) for \( S n_{i,0} \) in terms of \( N \), substitute into (c).

\[
\frac{dN}{dt} = 4\pi r_0^2 D_i \left[ (n_{i,\infty} - n_{i,0}) - n_{i,0} \frac{L}{R_i T} \frac{1}{4\pi r_0^2 D_H n c_p} \right]
\]

\[
\frac{dN}{dt} \left[ 1 + \frac{D_i L}{D_H R_i T c_p T n_i} \right] = 4\pi r_0^2 D_i (n_{i,\infty} - n_{i,0})
\]

**reduce growth rate**
4. Numbers \( \text{H}_2\text{O} \) on \( 0^\circ \) will show that even a low vapor pressure case is quick. Assume \( T = 200 \text{K} \).

\[
L (\text{vapor} - \text{solid}) = 680 \text{ cal g}^{-1} = 2.8 \times 10^6 \text{ J kg}^{-1} \\
R = 462 \text{ J K}^{-1} \text{ kg}^{-1}
\]

\[
\frac{L}{RT} = 30.3 \text{ (no units)}
\]

\[
P_s (200) = 0.16 \text{ Pa} \quad \text{[Lodders & Fegley]}
\]

\[
P_{\text{few}} = 6 \text{ mb} = 600 \text{ Pa}
\]

\[
N.B., \quad \frac{0.16 \text{ Pa}}{600 \text{ Pa}} \sim 2.6 \times 10^{-4} \sim 0.03 \%
\]

First conclusion: \( \text{H}_2\text{O} \) is in \( 0^\circ \) atm, and at least to order of magnitude is saturated.

Next, rewrite growth equation to give doubling time estimate

\[
\frac{1}{N} \frac{dN}{dt} = \frac{1}{1 + \frac{D_i (\frac{L}{RT})^2 R_i n_i \epsilon_p}{P}} - \frac{4\pi r_0 D_i (n_{i0} - n_i)}{4\pi^2 r_0^3 n_{i,\text{liquid}}}
\]

We know \( P_{\text{liquid}} = m \mu n_{\text{liquid}} = 10^3 \text{ kg m}^{-3} \)

so multiply through by \( m \mu = 1.6 \times 10^{-27} \times 18 \times 10^{-3} \),

Also, drop the \( L^2 \) term, \( \epsilon \) small, \( \Theta \) high important.

\[
\frac{1}{N} \frac{dN}{dt} \sim \frac{D_i}{r_0^2} \frac{P_{\text{oi}} - P_{\text{oi}}}{P_2}
\]

Rearrange: \( \frac{1}{N} \frac{dN}{dt} \sim \frac{D_i}{r_0^2} \frac{P_{\text{oi}} P_{\text{oi}}}{P_2} \frac{P_{\text{oi}} - P_{\text{oi}}}{P_{\text{oi}} - P_{\text{oi}}} \)
\[ n = \frac{6}{1000}, \eta_0 = \frac{6}{1000} \cdot 2.7 \times 10^{-25} \]

5. Consequence, \( D = \lambda C_s \), \( \lambda = \frac{1}{nA} \), \( A = 10^{-18} \text{ m}^2 \)

\[ \lambda = \frac{1}{nA} = \frac{1}{1.6 \times 10^{-23} \times 10^{-18}} = 6 \times 10^{-6} \text{ m} \]

The area at the base of \( \sigma \theta \) is \( \frac{P_0}{R T} = 0.016 \text{ kg m}^{-2} \)

\[ \frac{1}{N} \frac{dN}{dt} = \frac{\lambda C_s}{r_0^2} \cdot \frac{P_0}{\rho} \cdot \frac{P_0}{\rho} \cdot S \]

\[ S = \text{supersat} \]

\[ \frac{1}{N} \frac{dN}{dt} = \left( \frac{r_{\text{reference}}}{r} \right)^2 \frac{1}{52} \text{ s} \]

Assumption: Like \( \theta \) \( S \) never exceeds a few \( 10^3 \). We will discuss later.

\[ \frac{1}{N} \frac{dN}{dt} = \left( \frac{r_{\text{reference}}}{r} \right)^2 \frac{1}{52} \text{ s} \]

This is fast compared to a day or even to convective overturning.

Conclusion: If proper updraft exists, ice crystals on \( 0^\circ \) will grow well past \( 1 \mu \) very quickly.

What will stop them?
Fall rates

1) Viscous laminar regime, $\lambda \ll r$, $\frac{Wr}{v} \ll 1$

\[ mg = \rho \nu \frac{w}{r} \cdot r^2 \]

Force = Stress \cdot Area

Exact treatment gives

\[ W_F = \frac{2}{9} \frac{\rho \nu}{\nu} \frac{gr^2}{v} \]

2) $\lambda \gg r$ no viscosity - only impacts

Momentum dilation

\[ m \frac{dv}{dt} = \frac{\Delta M}{St} v \]

\[ \frac{\Delta M}{St} = c_n m A \]

\[ \therefore mg = cp r^2 w \]

Use $c\lambda = v$

\[ mg = \rho \nu \frac{w}{r} \cdot r^2 \]

Need transition $\frac{r^2}{\lambda}$ to $r$

\[ W = \frac{2}{9} \frac{\rho \nu}{\nu} \frac{gr^2}{v} (1 + \frac{1}{r}) \] Stokes-Cunningham

3) Large particles $F = \frac{r^2}{2} \rho W^2 = mg$

Measurements $\Rightarrow F = \frac{\pi}{2} r^2 W^2 \rho C_D$, $C_D \sim 0.45$
Miscellaneous Comments on Specific Cases

1. Thickness of condensate cloud.

Assume \( x = \text{mass fraction} \ll 1 \). Then in convecting layer

\[
T = T_0 - \frac{g}{c_p} z
\]

Saturation

\[
\frac{dp_a}{dT} = \frac{L}{R \nu} \frac{p_a}{T}
\]

\[
\ln \frac{p_a}{p_{a0}} = \frac{L}{R \nu} \left( \frac{1}{T_0} - \frac{1}{T} \right) \quad \text{use} \quad T_0 - T \ll T_0
\]

\[
= \frac{L}{R \nu T_0} (T - T_0) \sim \frac{L (-g z)}{R \nu T_0 c_p T_0}
\]

\[
p_a = p_{a0} \exp \left[ - \frac{L}{R \nu T_0 c_p} \frac{g z}{RT_0} \right]
\]

Scale height \( H = RT_0 / g \)

Saturned vapor scale height, \( p_a = p_{a0} e^{-z/H_V} \)

given by

\[
H_V = H_0 \cdot \frac{R \nu T_0}{L} \cdot \frac{c_p}{R} \ll H_0
\]

Expect relatively thin condensation layers.
2. Buoyancy

Consider steady ascent.

Follow panel

\[
\delta e = \delta u + \delta (\alpha p) + \delta g \\
\implies \delta e = \delta (u + \alpha p) - \alpha \delta p + \delta g
\]

\[
\frac{\text{mass vapor}}{\text{total mass}}
\]

Force equation

\[
\frac{D\mathbf{F}}{Dt} + \frac{1}{\rho} \nabla \rho = - \nabla \phi
\]

\[
\rho \frac{Du}{Dt} + (\nabla \times \mathbf{v}) \times \mathbf{v} + \frac{\nabla (\rho u^2)}{2} + \frac{1}{\rho} \nabla p + \nabla \phi = 0
\]

Assume steady, dot with \( \mathbf{v} \), resolve along streamline,

\[
S \left( \frac{\rho u^2}{2} \right) + \alpha \delta p + \delta \phi = 0
\]

Bernoulli! Then substitute \( \alpha \delta p \), \( \to \)

\[
c_p T + \phi + L_q + \frac{v^2}{2} = \text{constant}
\]

Called static energy or moist static energy.

One important result: rise through cloud yields \( c_p \Delta T = L_q \) warming.

Dry

\[
\downarrow \quad \text{ascent} \quad \Delta T = \frac{L_q}{c_p}
\]

\[
\text{at base}
\]
3. Optical depth

\[ \tau = 1 \implies \text{particles along path cover area} \]

\[ \Rightarrow A \cdot n \cdot l = 1 \quad A = \text{area of particle} \]

Define in general \( \tau = A n l \), \( 0 \leq \tau < \infty \)

Evaluate \( n \) from mass of saturated vapor and an assumed particle radius.

\[ n = \frac{P_s}{m_p} = \frac{P_s}{\frac{4}{3} \pi r^3} \]

\[ \text{Table for } H_2 O, \quad r = 10^{-6} \text{ m} \]

<table>
<thead>
<tr>
<th></th>
<th>( \Delta T )</th>
<th>( H )</th>
<th>( H_v )</th>
<th>( n )</th>
<th>( l (\tau=1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ψ</td>
<td>1 bar, 220 K</td>
<td>.005 K</td>
<td>6 km</td>
<td>.4 km</td>
<td>2e8</td>
</tr>
<tr>
<td>Φ</td>
<td>1 bar, 280 K</td>
<td>15 K</td>
<td>8 km</td>
<td>1.5 km</td>
<td>4e12</td>
</tr>
<tr>
<td>Σ</td>
<td>6 hPa, 200 K</td>
<td>0.3 K</td>
<td>10 km</td>
<td>1.0 km</td>
<td>4e8</td>
</tr>
<tr>
<td>Ψ</td>
<td>7 bar, 280 K</td>
<td>2.8 K</td>
<td>40 km</td>
<td>7 km</td>
<td>2e12</td>
</tr>
</tbody>
</table>

In addition, \( \frac{\Delta m}{m} = 1.1 \times 10^{-4} \)

Comments: Small vapor scale heights

Large latent heat \( \Phi \) 4

Large \( \frac{\Delta m}{m} \) 4

Too large \( n \) \( \Phi \) 7 \( \Rightarrow \) growth