Astro 523 Signal Processing

Fishing for Planets Around Pulsars

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May, 2001

Introduction

The first planetary system outside our own was detected around the millisecond pulsar PSR B1257+12 (Wolszczan and Frail, 1992). The millisecond pulsars are believed to be old neutron stars spun up through mass accretion during an X-ray binary phase. It is thought that the accretion disk left as a result of this process might give birth to planets.

In principle, a planet may survive the supernova explosion leading to the formation of the neutron star, provided that the explosion is asymmetric. However, the odds for this event seem to be extremely low. More likely is the re-formation of planets after the explosion, from the infalling debris.

There are different methods that can be used to search for planets around pulsars. By far the most common is the least squares fit, whence one attempts to fit planetary elements to pulsar timing observations. However, Konacki and Maciejewski propose a new technique based on the normalized Lomb-Scargle periodogram.[5] This method is based on analyzing the spectrum of the time of arrival residuals, and looking for quasi-periodicity. The strength of this procedure is that it allows for determination of the orbital parameters of planets, and is very sensitive. In this report we will concentrate on this technique.

Time of Arrival Residuals

In this section we will derive the expected quasi-periodic form of the time of arrival residuals. We will closely follow the derivation presented in Konacki and Maciejewski.[5]

Assume that there are $N$ planets around the pulsar. It is convenient to work in the center of mass (or barycentric) reference frame of the system shown in fig.1. The $XY$ plane is the plane of the sky such that the $X$ axis points towards north. The $Z$ axis is directed towards the barycenter of the solar system, i.e. the revolution of the earth around the sun is subtracted. $\mathbf{R}_*\,$ denotes the position of the pulsar, and $\mathbf{R}_j\,$ denotes the position of the $j$th planet. The orbits of the individual planets are described by the Keplerian parameters, namely by $\{T_p, a, e, i, \omega, \Omega\}$, or by the time of the pericenter, semimajor axis, eccentricity,
By definition, we have,

\[ R_\star = -\frac{1}{M_\star} \sum_{j=1}^{N} m_j R_j \quad (1) \]

where \( M_\star \) is the mass of the pulsar and \( m_j \) is the mass of the \( j \)th planet.

The pulse time of arrival residue due to the motion of the pulsar around the center of mass as observed from the solar system barycenter is given by the following formula,

\[ \Delta t = -\frac{1}{c} R_\star \cdot \hat{Z} \quad (2) \]

where \( \hat{Z} \) is the unit vector along the \( Z \) axis and \( c \) is, as usual, the speed of light. This formula should not be confusing; it simply gives the excess of time it takes for the pulse to traverse
the additional distance due to the current position of the pulsar. Note that if the pulsar is further away from us, the scalar product is negative and the time residue is positive, as one would naturally expect.

Now, if we substitute the first equation into the second,

$$\Delta t = \frac{1}{cM_*} \sum_{j=1}^{N} m_j R_j \cdot \hat{Z} = \frac{1}{cM_*} \sum_{j=1}^{N} m_j Z_j$$

(3)

where obviously \(Z_j\) is the \(Z\) coordinate of the \(j\)th planet.

Assume that all planets move in elliptic orbits. We will take the following formulae for granted, and leave the derivations as an exercise to the interested reader. The radius vector of a planet moving in an elliptic orbit can be expressed as (see, for instance, Brumberg 1980)

$$R(t) = Pa(\cos E(t) - e) + Q a \sqrt{1 - e^2} \sin E(t)$$

(4)

where,

\[ P = I \cos \omega + m \sin \omega, \quad Q = -I \sin \omega + m \cos \omega \]

(5)

\[ I = \begin{pmatrix} \cos \Omega \\ \sin \Omega \\ 0 \end{pmatrix}, \quad m = \begin{pmatrix} -\cos i \sin \Omega \\ \cos i \cos \Omega \\ \sin i \end{pmatrix} \]

(6)

The so-called eccentric anomaly \(E(t)\) is given as an implicit function of time through the Kepler equation,

$$E - e \sin E = \mathcal{M}$$

(7)

where \(\mathcal{M}\) is the mean anomaly, given by

$$\mathcal{M} = n(t - T_p), \quad n = \frac{2\pi}{P}$$

(8)

and \(P\) is the period of a planet. Then, by definition, \(n\) is the angular frequency (not to be confused with \(\omega\)). Since functions \(\cos E\) and \(\sin E\) are periodic with respect to \(\mathcal{M}\) they can be expanded in the Fourier series (see Aksenov 1986)

$$\cos E = -\frac{1}{2} e + \sum_{k \in \mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \cos(k\mathcal{M})$$

(9)

$$\sin E = \sum_{k \in \mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \sin(k\mathcal{M})$$

(10)

where \(J_n(z)\) is the Bessel function of the first kind of order \(n\). \(\mathbb{Z}_0\) denotes the set of all positive and negative integers (in other words, zero is excluded). Note that the order of

\footnote{Incidentally, the vector equations can be combined in a special kind of reduced matrix equation,

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} I \\ m \end{pmatrix}$$

but we will not make use of these in the near future.}
Bessel functions in the above formula can be negative. In fact, Bessel functions with negative orders are mathematically independent functions from positive orders, unless that order is an integer.\[1\] This property will be of use soon.

Substituting the above expansions into eq.4 we receive,

\[
\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{A} \sum_{k \in \mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \cos(k\mathcal{M}) + \mathbf{B} \sum_{k \in \mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \sin(k\mathcal{M})
\]

where,

\[
\mathbf{R}_0 = -\frac{3}{2} \mathbf{P} a e, \quad \mathbf{A} = \mathbf{P} a, \quad \mathbf{B} = \mathbf{Q} a \sqrt{1 - e^2}
\]

This result can also be expressed in complex notation as,

\[
\mathbf{R}(t) = \mathbf{R}_0 + \sum_{k \in \mathbb{Z}_0} \Theta_k e^{ik\mathcal{M}}
\]

where,

\[
\Theta_k = \frac{1}{2k} \left[ \mathbf{A}(J_{k-1} - J_{k+1}) - i \mathbf{B}(J_{k-1} + J_{k+1}) \right]
\]

The derivation of this relation is rather trivial, but in order to clarify the implications of this notation it is useful to go through the steps. However, this will be done in the appendix. Here we will just quote our results.

Note that the summation in the above formula is carried over both positive and negative integers. It turns out that (see the appendix) by dropping the factor of 1/2 the series can be converted into a summation over only positive integers. Apparently, this is much more convenient for programming purposes. Incidentally, only the real part is of interest.

Using the definition of \(\mathcal{M}\) given by eq.8, we thus obtain the Fourier expansion of \(\mathbf{R}(t)\) in the time domain,

\[
\mathbf{R}(t) = \mathbf{R}_0 + \sum_{k \in \mathbb{Z}_0} \Lambda_k e^{ik\mathcal{M}t}, \quad \text{where} \quad \Lambda_k = \Theta_k e^{-iknT_p}
\]

This expansion has an important property: it is possible to prove, using properties of Bessel functions, that the terms in the expansion are monotonically decreasing (in magnitude). However, we will not divert into unnecessary detail, and will instead leave this as yet another exercise for the reader. It is perhaps worth noting that the sole motivation behind the manipulation of the series representation is to attain this monotonic decrease, apart from the joy of playing around with our most beloved Bessel functions. The decay of coefficients in the expansion assures that the first few terms will be sufficient to carry out the calculations up to a desired accuracy.

The expansion for the \(Z\) component of the radius vector of the \(j\)th planet is then given by,

\[
Z_j(t) = Z_{j0} + \sum_{k \in \mathbb{Z}_0} \Phi_{jk} e^{ik\mathcal{M}_j t}
\]

where, using the definitions for the various parameters (given through eqs.5, 6 and 12), we have

\[
Z_{j0} = -\frac{3}{2} e_j a_j \sin i_j \sin \omega_j \quad (17)
\]

\[
C_j = a_j \sin i_j \sin \omega_j \quad (18)
\]

\[
S_j = -a_j \sqrt{1 - e_j^2} \sin i_j \cos \omega_j \quad (19)
\]
and,
\[ \Phi_{jk} = \frac{1}{2k} \left[ C_j(J_{k-1} - J_{k+1}) + iS_j(J_{k-1} + J_{k+1}) \right] e^{-ikn_jT_p} \] (20)

Substituting the Z component into the residue equation, eq. 3 we get,
\[ \Delta t(t) = \sum_{j=1}^{N} \tilde{m}_j Z_{j0} + \sum_{j=1}^{N} \sum_{k} \tilde{\Phi}_{jk} e^{ikn_j t} \] (21)

with the following two new definitions,
\[ \tilde{m}_j = \frac{1}{c M_*} m_j, \quad \tilde{\Phi}_{jk} = \tilde{m}_j \Phi_{jk} \] (22)

We have thus developed the basic formulation for the time of arrival residues. By using the definitions above (in particular, by using the last six equations and eq. 8 for \( n_j \)), it is possible to generate sample data which can subsequently be analyzed by spectral methods.

**Simulation of Data**

In the simulation we use orbital data for three planets discovered around pulsar PSR B1257+12 (see Wolszczan, 1994). The data is listed in the table below.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>0.0</td>
<td>0.0182</td>
<td>0.0264</td>
</tr>
<tr>
<td>( T_p ) (JD)</td>
<td>2,448,754.3</td>
<td>2,448,770.3</td>
<td>2,448,784.4</td>
</tr>
<tr>
<td>( P ) (days)</td>
<td>25.34</td>
<td>66.540</td>
<td>98.220</td>
</tr>
<tr>
<td>( \omega ) (°)</td>
<td>0.0</td>
<td>249</td>
<td>106</td>
</tr>
<tr>
<td>( m (M_\oplus) )</td>
<td>0.015/sin ( i_A )</td>
<td>3.4/sin ( i_B )</td>
<td>2.8/sin ( i_C )</td>
</tr>
<tr>
<td>r (AU)</td>
<td>0.19</td>
<td>0.36</td>
<td>0.47</td>
</tr>
</tbody>
</table>

Note that we do not have to bother about the inclination angle, since this information is already included in the masses of the individual planets. In other words, the masses of the planets are measured in units of \( M_\oplus/\sin i_j \). This results in the simplification of the above formulae. Also note that the time of the pericenter, \( T_p \) is given in terms of the Julian date, which is a continuous count of days since noon (GMT) on January 1, 4713 B.C.\(^2\)

We assume that the mass of the pulsar is approximately a solar mass, i.e.
\[ M_* \sim M_\odot \approx 2 \times 10^{30} \text{kg} \] (23)

In terms of earth masses, \( M_\oplus \approx 6 \times 10^{24} \text{kg} \) this is,
\[ M_* \sim \frac{1}{2} 10^6 M_\oplus \] (24)

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\(^2\)"Why the year 4713 B.C.? The Julian Day system of numbering the days is a continuous count of days elapsed since the beginning of the Julian Period. This period was devised by Joseph Justus Scaliger, a French classical scholar in the 16th century. Scaliger calculated the Julian Period by multiplying three important chronological cycles: the 28-year solar cycle, the 19-year lunar cycle, and the 15-year cycle of tax assessment called the Roman Indiction. To establish a beginning point for his Julian Day system, Scaliger calculated the closest date before 1 B.C. which marked the first day for the beginning of all three cycles. This day is January 1, 4713 B.C., which is Julian Day number 1." (from the Chandra X-ray Observatory webpage hosted by Harvard.)
Since the semimajor axis is expressed in astronomical units, AU $\approx 1.5 \times 10^8$km we also choose to express the speed of light in appropriate units,

$$c \approx 3 \times 10^8 \text{km/s} \approx 2 \times 10^{-6} \text{AU/ms}$$

The time of arrival residuals then have units of milliseconds,

$$\Delta t \sim \frac{a}{c M^*}$$

To simulate more realistic data, it is possible to add noise and introduce unequal sampling, as well.

Other orbital data is also available in the literature. For instance, Shabanova reports on a planet around pulsar PSR B0329+54.[13] However, Konacki et al. using the method to be outlined in the following section, recently showed that the variations in the pulse of this particular pulsar are unlikely to be due to the presence of planets.[8]

Finally, it is worth noting that these pulsars have been extensively studied by J. M. Cordes.[4]

**Looking for Planets**

We are now ready to explore ways to discover the planets that we hid around our pulsar in the previous section.

One of the most commonly applied procedures for finding planets around pulsars is least squares fitting. This was the method that eventually led to the discovery of the first planetary system around a pulsar, namely PSR B1257+12. Other methods exist as well. Making use of genetic algorithms and the normalized Lomb-Scargle periodogram are two contemporary methods. In this paper we will employ the latter. Yet, it might be interesting to go through filtering techniques first. In that respect the new edition of Press et al. proves out to be a very useful reference.

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3If we choose units such that the speed of light is unity, then $M^* \sim 1M_{\oplus} \text{AU/ms}$. That is what we have done in our program.

4The author finds it rather convenient and proper to make reference to Cordes at this point, who will eventually grade this report.
Optimal Filtering and Periodograms

Assume that we want to measure a signal $u(t)$. However, the measurement will yield a corrupted signal $c(t)$ due to instrumental imperfections or environmental intrusions. The first will be detected as an imperfect delta-function (or impulse) response of the apparatus, and is equivalent to convolving the original signal with some known response function $r(t)$,

$$s(t) = \int_{-\infty}^{\infty} r(t-\tau) u(\tau) d\tau \quad \text{or} \quad S(f) = R(f) U(f)$$  \hspace{1cm} (27)

where $S, R, U$ are the Fourier transforms of $s, r, u$, respectively.

The second will appear as an additional noise $n(t)$, so that the measured (corrupted) signal will be,

$$c(t) = s(t) + n(t) \quad \text{or} \quad C(f) = S(f) + N(f)$$  \hspace{1cm} (28)

Note that the presence of noise complicates the situation quite a bit. In its absence it is possible to retrieve the original signal through deconvolution, i.e. by simply dividing $C(f)$ by $R(f)$. However, when noise is present, one has to devise an optimal filter $\phi(t)$ or $\Phi(f)$, which, when convolved with the measured signal and then deconvolved with the response yields a signal $\tilde{u}(t)$ or $\tilde{U}(f)$ that is as close as possible to the uncorrupted signal. In other words, the true signal $U(f)$ is estimated by,

$$\tilde{U}(f) = \frac{C(f) \Phi(f)}{R(f)}$$  \hspace{1cm} (29)

The estimate is close to the real signal in the sense that,

$$\int_{-\infty}^{\infty} |\tilde{u}(t) - u(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{U}(f) - U(f)|^2 df$$  \hspace{1cm} (30)

is minimized. Using the above relations in the right hand side and minimizing the result

![Figure 4: Optimal (Wiener) filtering.][11]

with respect to $\Phi$, we get

$$\Phi(f) = \frac{|S(f)|^2}{|S(f)|^2 + |N(f)|^2}$$  \hspace{1cm} (31)
This gives the optimal filter.

Note that since signal and noise are uncorrelated we have,

$$|C(f)|^2 \approx |S(f)|^2 + |N(f)|^2$$

for the spectral density. The resulting plot usually has a form similar to that shown in fig.4. In that case, information about the shapes of the signal and noise can be extracted straightaway. It turns out that the errors in the results of the optimal filter are second order in the precision of the filter, i.e. even a crude estimate of the filter may give surprisingly good results.

The above method of power spectrum estimation is a simple version of a periodogram. The inverse is also true: a periodogram is an estimate of the power spectrum (or power spectral density).

Consider evenly sampled data, $c_j = h(j \Delta)$ where $\Delta$ is the sampling interval and its reciprocal is the sampling rate. Then the Nyquist (critical) frequency is, by definition,

$$f_c \equiv \frac{1}{2\Delta}$$

The sampling theorem states that a sampled data set of the above form contains complete information about all spectral components in a signal $c(t)$ up to the Nyquist frequency, and aliased information about higher frequencies.

The FFT of the signal is given by,

$$C_k = \sum_{j=0}^{N-1} c_j e^{2\pi i j k/N} \quad k = 0, \ldots, N - 1$$

Then the periodogram estimate of the power spectrum is defined at $N/2 + 1$ frequencies as,

$$P(0) = P(f_0) = \frac{1}{N^2} |C_0|^2$$

$$P(f_k) = \frac{1}{N^2} \left[ |C_k|^2 + |C_{N-k}|^2 \right] \quad k = 1, 2, \ldots, \left( \frac{N}{2} - 1 \right)$$

$$P(f_c) = P(f_{N/2}) = \frac{1}{N^2} |C_{N/2}|^2$$

where $f_k$ is defined only for the zero and positive frequencies,

$$f_k \equiv \frac{k}{N\Delta} = 2f_c \frac{k}{N} \quad k = 0, 1, \ldots, \frac{N}{2}$$

Note that the periodogram is normalized, i.e. the sum of the $N/2 + 1$ values of $P$ is equal to the mean squared amplitude of the signal $c_j$,

$$P(f_0) + \sum_{k=1}^{N/2-1} P(f_k) + P(f_{N/2}) =$$

$$= \frac{1}{N^2} \left[ |C_0|^2 + \sum_{k=1}^{N/2-1} |C_k|^2 + \sum_{k=1}^{N/2-1} |C_{N-k}|^2 + |C_{N/2}|^2 \right]$$

$^5$The relation is exact when integrated over all frequencies.
\[
\frac{1}{N^2} \left[ |C_0|^2 + \sum_{k=1}^{N/2-1} |C_k|^2 + |C_{N/2}|^2 + \sum_{k=N/2+1}^{N-1} |C_k|^2 \right]
\]

\[
= \frac{1}{N^2} \sum_{k=0}^{N-1} |C_k|^2 = \frac{1}{N} \sum_{j=0}^{N-1} |C_j|^2
\]

where we have made use of Parseval’s theorem in discrete form,

\[
\sum_{j=0}^{N-1} |c_j|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |C_k|^2
\]

We are now left with the question of how accurate the periodogram estimate is with respect to the true spectrum. There are two important problems with periodograms. The first is leakage, whence power leaks into neighboring frequency bins. The problem can be resolved by data windowing. This method consists of multiplying the input \(c_j\) by a window function \(w_j\) that changes gradually from zero up to a maximum and then back to zero, as \(j\) ranges from 0 to \(N\).\(^6\)

The second problem with the periodogram estimate (as written in the above form) is that it does not become more accurate as the number of data points, \(N\), goes to infinity. Fortunately, this problem can also be resolved rather easily.

**The Normalized Lomb-Scargle Periodogram Method**

FFT methods are applicable to evenly sampled data sets. However, in real situations, one quite frequently has to deal with unevenly sampled data. It might be either due to instrumental drop-outs or simply due to observational constraints.

One obvious solution to this problem is to interpolate the real data to evenly sampled data. While this might be an adequate resolution for a few missing data points, it will most likely perform rather poorly in the case of large gaps in the data, as is the usual case in astronomy. Such long data gaps can produce spurious bulges of power at low frequencies, roughly corresponding to wavelengths of order of the gap size.

A special method is required to deal with unevenly sampled data sets. One such method, proposed by Lomb and later elaborated by Scargle, is the normalized Lomb-Scargle periodogram (the NLSP). Suppose that there are \(N\) data points \(h_i(t_i), i = 1, \ldots, N\). We shall first calculate the mean and the variance of the data,

\[
\bar{h} \equiv \frac{1}{N} \sum_{i=1}^{N} h_i, \quad \sigma^2 \equiv \frac{1}{N-1} \sum_{i=1}^{N} (h_i - \bar{h})^2
\]

Then the NLSP, or the spectral power as a function of angular frequency \(\omega \equiv 2\pi f > 0\), is defined as,

\[
\begin{align*}
P_N(\omega) &\equiv \frac{1}{2\sigma^2} \left\{ \frac{\left[ \sum_j (h_j - \bar{h}) \cos \omega(t_j - \tau) \right]^2}{\sum_j \cos^2 \omega(t_j - \tau)} + \frac{\left[ \sum_j (h_j - \bar{h}) \sin \omega(t_j - \tau) \right]^2}{\sum_j \sin^2 \omega(t_j - \tau)} \right\} \\
\end{align*}
\]

\(^6\)Here we quote Press et al. on this matter: “There is a lot of perhaps unnecessary lore about choice of a window function, and practically every function that rises from zero to a peak and then falls again has been named after someone.”\(^{[11]}\)
Here $\tau$ is defined through the following equation,

$$\tan 2\omega \tau = \frac{\sum_j \sin 2\omega t_j}{\sum_j \cos 2\omega t_j}$$  \hspace{1cm} (41)

The term “normalized” refers to the $\sigma^2$ appearing in the denominator.

The constant $\tau$ acts like some kind of offset that makes the power spectrum completely independent of any constant shift of all $t_j$’s. This particular choice of the offset makes the periodogram equivalent to linear least squares fitting to the model,

$$h(t) = A \cos \omega t + B \sin \omega t$$  \hspace{1cm} (42)

The NLSP has the strength that it weighs the data on a “per point” basis rather than on a “per time interval” basis.

We would like to know the significance of peaks in the power spectrum, when the signal is contaminated by an independent (white) Gaussian noise. The null hypothesis is that the data points are independent Gaussian random values. Scargle shows that $P_N(\omega)$ has an exponential probability with unit mean.\[12\] That is, the probability that $P_N(\omega)$ will be between some positive $z$ and $z + dz$ is $\exp(-z)dz$. Consequently, the probability of having a value up to a certain $z$ is then,

$$P(\leq z) = \int_0^z e^{-z}dz = 1 - e^{-z}$$  \hspace{1cm} (43)

If we scan $M$ independent frequencies, the combined probability that none gives values larger than $z$ is simply,

$$P(\leq z) = (1 - e^{-z})^M$$  \hspace{1cm} (44)

It then follows that, the false alarm probability of the null hypothesis is,

$$P(> z) = 1 - (1 - e^{-z})^M$$  \hspace{1cm} (45)

This is also a measure of the significance level of the peaks. A small value indicates a highly significant peak (or periodic signal). Practical values are 0.05 or less.

It is therefore important to know $M$, the number of independent frequencies. In general, $M$ depends on the number of sampled frequencies, the total number of data points $N$, and their detailed spacing. It turns out that $M$ is approximately equal to $N$ in the case of even sampling.

**Application of the NLSP to the Problem**

We are now in a position to discuss the implementation of the NLSP to the search of planets around pulsars, as proposed by Konacki and Maciejewski.[5]

Assume that the time of arrival (TOA) residuals are due to the presence of planets. The larger planets can be detected through the NLSP by simply observing the significant peaks in the spectrum. These frequencies (or first harmonics) are then subtracted from the signal. This might be accomplished, for instance, by the method of Ferraz-Mello (1981). The remaining signal is then re-processed by the NLSP in the search for higher order or new harmonics. Higher order harmonics (i.e. linear combinations of the first harmonics) are still due to the same planets, while the presence of new harmonics implies the discovery of an
additional planet. The procedure is then repeated until no significant peaks remain in the spectrum.

Due to severe time limitations we will not go deeper than the first step, namely the application of the NLSP to the simulated time of arrival residuals. To this end, we make use of the Fortran routines presented in Press et al.

Results and Conclusions

We performed the NLSP on the generated data. The following few figures show some of the results. In all of them the horizontal axis is frequency in days$^{-1}$, and the vertical axis is the power spectral density. The frequency $f$ of the most significant peak in each spectrum is given in parentheses following the caption of the corresponding figure, together with the significance level $P$ of this peak.

In fig.5 we have plotted the spectra for evenly spaced data sets of 200 and 400 points, respectively. The time of arrival residuals that serve as the input signal are the same as those plotted in fig.3.

![Figure 5: The NLSP applied to the simulated time of arrival residues. (For the left figure $f = 0.01026$, $P = 6.3 \times 10^{-22}$, and for the right figure $f = 0.01023$, $P = 1.3 \times 10^{-45}$.)](image)

Note that there are two peaks, one at frequency 0.0102 days$^{-1}$ and one at approximately 0.015 days$^{-1}$. These frequencies correspond to two planets with periods of 98 and 67 days, respectively. Remember that the true periods of the three planets that we have included in our simulation are 25.34, 66.54 and 98.22 days. We have thus been able to retrieve planets $\mathcal{B}$ and $\mathcal{C}$, while planet $\mathcal{A}$ has eluded us in our first attempt. Yet, Konacki and Maciejewski were able to find the last planet after subtracting the first and second harmonics of planets $\mathcal{B}$ and $\mathcal{C}$.[5] However, we choose to stop at this point.

Figs.6 through 8 show the effects of missing data points and the presence of large gaps in the data. In all cases, the peaks are clearly distinguishable and are at the right frequencies.

We thus see that the NLSP method can indeed reveal the presence of planets around pulsars. Yet, this remains to be validated through subsequent steps involving filtrations of harmonics, as discussed before. However, this is beyond the scope of this report and is left as one final exercise to the reader.
Figure 6: The effect of missing data. 99 points of a total of 200 have been randomly removed from the input array. In a way, this is unevenly spaced data. ($f = 0.01015, P = 1.7 \times 10^{-8}$)
Figure 7: 80 more points have been randomly deleted from the input. There seems to be a bias in the selection mechanism towards the later points. A peak is still visible at the expected frequency. \((f = 0.01001, P = 0.047)\)
Figure 8: The effect of a large gap in the input signal. 50 consecutive data points are missing in a total of 200. \( f = 0.01026, P = 4.8 \times 10^{-16} \)
Appendix: Bessel Functions

In this section, we will provide the steps leading from eq.11 to eq.13. While these steps are rather trivial and nothing but mathematical manipulations, they are of importance for understanding the resulting formulae.

We will make use of a few relations from Abramowitz.[1] First, for integer orders the following relation exists between negative and positive orders,

$$J_{-\nu} = (-1)^\nu J_{\nu}$$ (46)

We assume without loss of generality that the order $\nu$ is a positive integer, and let’s denote the set of all positive integers by $N_0$. Second, Bessel functions can be expanded in a series of the following form,

$$J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\Gamma(\nu + k + 1)}$$ (47)

This formula simply implies that,

$$J_\nu(-z) = (-1)^\nu J_\nu(z)$$ (48)

We will try to derive an alternative representation of the series that appear in eq.11. In particular, note that the following series can also be represented as,

$$\sum_{k\in\mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \cos(kM) = \sum_{\nu\in N_0} \frac{1}{\nu} J_{\nu-1}(\nu e) \cos(\nu M)$$

$$+ \sum_{\nu\in N_0} \frac{1}{\nu} (-1)^\nu J_{-\nu-1}(\nu e) \cos(-\nu M)$$

Using the above relations for Bessel functions, the second term on the right hand side reduces to,

$$- \sum_{\nu\in N_0} \frac{1}{\nu} J_{\nu+1}(\nu e) \cos(\nu M)$$

whence,

$$\sum_{k\in\mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \cos(kM) = \sum_{\nu\in N_0} \frac{1}{\nu} \left[ J_{\nu-1}(\nu e) - J_{\nu+1}(\nu e) \right] \cos(\nu M)$$ (49)

This representation involves only positive integers. Note that the right hand side remains unchanged under the substitution of $-\nu$. This allows us to rewrite the equation as,

$$\sum_{k\in\mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \cos(kM) = \sum_{k\in\mathbb{Z}_0} \frac{1}{2k} \left[ J_{k-1}(ke) - J_{k+1}(ke) \right] \cos(kM)$$ (50)

A similar expression can be found for the second term in eq.11,

$$\sum_{k\in\mathbb{Z}_0} \frac{1}{k} J_{k-1}(ke) \sin(kM) = \sum_{k\in\mathbb{Z}_0} \frac{1}{2k} \left[ J_{k-1}(ke) + J_{k+1}(ke) \right] \sin(kM)$$ (51)

This completes the derivations.
Computer Implementation of Bessel Functions

Press et al. provide a perfect routine for calculating Bessel functions of positive integer order. It is therefore important to convert the summations such that only positive integer orders are involved. It is also highly recommended that double precision be used.
References


