Fourier Transforms
**LSI Systems**

In the notes *Linear, Shift-invariant Systems and Fourier Transforms* on the course website it is shown that exponentials are an appropriate basis for LSI systems. LSI systems involve an impulse response (equivalent to a Green’s function), $h(t)$ that is convolved with the input of the system to obtain the output. There are then **two basic kinds of systems** if we consider them to involve time-domain quantities:

**Causal**

$h(t) = 0$ for $t < 0$

The output depends only on past values of the input.

$$H(s) = \int_{0}^{\infty} dt' e^{-st'} h(t') \quad \text{Laplace transform}$$

**Acausal**

$h(t)$ not necessarily $0$ for $t < 0$

$$H(s) = \int_{-\infty}^{\infty} dt' e^{-st'} h(t')|_{s=\omega} \quad \text{Fourier transform}$$

Exponentials are useful for describing the action of a linear system because they are eigenfunctions of the system. If we can describe the actual input function in terms of exponential functions, then determining the resultant output becomes trivial. This is, of course, the essence of Fourier transform treatments of linear systems and their underlying differential equations.

**Focus on Fourier Transforms**

We will use only FTs because we will use them for spatial (and other) domain analyses. Also, with digital data sets the notion of past and future becomes blurred unless computations are being done in real time.

There are **three classes of Fourier Transforms** available whose utility depends on the nature of the quantity being considered.

**Fourier Transform (FT):** applies to continuous, aperiodic functions:

$$x(t) = \int_{-\infty}^{\infty} df \ e^{2\pi ft} \tilde{X}(f)$$
\[ X(f) = \int_{-\infty}^{\infty} dt \, e^{-2\pi i ft} x(t) \]

The basis functions \( e^{2\pi if t} \) are orthonormal on \([-\infty, \infty]\)

\[ \int_{-\infty}^{\infty} dt \, e^{2\pi if t} e^{-2\pi if t} = \delta(t) \]

**Fourier Series**: applies to continuous, periodic functions with period \( P \); discrete frequencies:

\[ x(t) = \sum_{n=0}^{\infty} e^{2\pi i (n/P)t} \hat{X}_n \]

\[ \hat{X}_n = \frac{1}{P} \int_0^P dt \, e^{-2\pi i (n/P)t} x(t) \]

\( x(t) \) periodic with period \( P \), orthonormality on \([0, P]\):

\[ \int_0^P dt \, e^{2\pi i (n/P)t} e^{-2\pi i (n'/P)t} = \delta_{nn'} \]

**Discrete Fourier Transform (DFT)**: applies to discrete time and discrete frequency functions:

\[ x_k = \sum_{n=0}^{\infty} e^{2\pi i nk/N} \hat{X}_n \]

\[ \hat{X}_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{-2\pi i nk/N} x_k \]

\( x_k, \hat{X}_n \) periodic with period \( N \), orthonormality on \([0, N]\):

\[ \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i nk/N} e^{-2\pi i nk'} = \delta_{kk'} \]

*The Fourier transform is the most general of these because aperiodic functions are the most general and therefore the other two can be derived from it.*

*Note that the DFT is not “just” a sampled version of the FT. Nontrivial consequences take place upon digitization, as we shall see.*
Attached is a table of Fourier transform theorems for continuous functions. These are enormously useful because they form the basis for particular algorithms and they also allow you to do complicated manipulations without explicitly doing any integrals.

1. Demonstrate the first two entries in the table by using a Gaussian function, taking its limit as its width $\to 0$. Start with the classic FT pair for a Gaussian function:

$$e^{-\pi t^2} \iff e^{-\pi f^2}.$$  

Construct a Gaussian in time $g(t)$, using only the scale and shift theorems in the table (no integrals!), which in the limit of an infinitesimal width, acts as a Dirac $\delta$-function, namely

$$f(t_0) = \int dt \, f(t) \delta(t - t_0),$$

where $f(t)$ is an arbitrary function. You need to pay attention to normalization of the function (e.g. unit area). *Without doing any integrals* investigate the Fourier transform of your Gaussian representation of the $\delta$-function. What is its shape and amplitude? A sketch is always helpful. What is the phase of the FT of $\delta(t - t_0)$ for an arbitrary time shift, $t_0$?

2. Prove the remaining theorems in the table, down to (and including) the integration theorem.

3. What does the integration theorem imply for the high-frequency components of the FT of the original function? Same question for the derivative theorem and the low-frequency components.

4. Derive the convolution and Parseval’s theorem for the DFT.

5. Analyze a time series by using discrete samples of the continuous-time model

$$x(t) = Ae^{2\pi ft} + n(t).$$

Let the sample interval be $\delta t = 1$ s. Separately consider frequencies corresponding to periods $P_0 = 1/f_0 = 10$ s, 50 s, and 0.1333 s and use a total time span of $T = 256$ s. The instructor will put example python code on the course web site that you can use or refer to if you like.

(a) Before doing anything numerically, calculate the discrete frequency bin $k$ in which you expect each sinusoid to be in or near.

(b) Numerically simulate the time series with your choice of amplitude $A$ and type of noise $n$. That is, generate noise samples that are statistically independent but with an amplitude distribution that is non-Gaussian of any type you like.

(c) For each time series calculate the power spectrum from the DFT as given in class. You will want to plot each spectrum. Explain your results.

(d) Calculate a histogram of the spectrum values and explain the results.

(e) Test Parseval’s theorem using your dual-domain results.
1D Fourier Transform Theorems

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\delta(f)$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$\delta(t)$</td>
</tr>
<tr>
<td>$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta)$</td>
<td>$\tilde{S}(f) = \Delta^{-1} \sum_{-\infty}^{\infty} \delta(f - k/\Delta)$ Bed of nails function</td>
</tr>
<tr>
<td>$y(t) = x(t) * h(t)$</td>
<td>$\tilde{X}(f)\tilde{H}(f)$ Convolution</td>
</tr>
<tr>
<td>$C_x(\tau) \equiv \int dt x^*(t)x(t + \tau)$</td>
<td>$</td>
</tr>
<tr>
<td>$x(t - t_0)$</td>
<td>$e^{-i\omega t_0} \tilde{X}(f)$ Shift theorem</td>
</tr>
<tr>
<td>$e^{+i2\pi f_0 t}x(t)$</td>
<td>$\tilde{X}(f - f_0)$ Shift theorem</td>
</tr>
<tr>
<td>$x(at)$</td>
<td>$a^{-1}\tilde{X}(f/a)$ Scaling theorem</td>
</tr>
<tr>
<td>$\tilde{X}(t)$</td>
<td>$x(-f)$ duality theorem</td>
</tr>
<tr>
<td>$x^*(t)$</td>
<td>$\tilde{X}^*(-f)$ Conjugation</td>
</tr>
<tr>
<td>$x^*(t) = x(t)$</td>
<td>$\tilde{X}^*(-f) = \tilde{X}(f)$ Hermiticity</td>
</tr>
<tr>
<td>$\int_{-\infty}^{\infty} dt</td>
<td>x(t)</td>
</tr>
<tr>
<td>$dx \quad dt$</td>
<td>$2\pi i f \tilde{X}(f)$ Derivative theorem</td>
</tr>
<tr>
<td>$\int dt' X(t')$</td>
<td>$(2\pi i f)^{-1} \tilde{X}(f)$ Integration theorem</td>
</tr>
<tr>
<td>$x(t) = \sum_m x_m \frac{\sin 2\pi f(t - m\delta t)}{2\pi f(t - m\delta t)}$</td>
<td>$\sum_m x_m e^{-2\pi imf\delta t} \Pi\left(\frac{f}{2\Delta f}\right)$ Sampling theorem</td>
</tr>
<tr>
<td>$\Pi(x) = \text{rectangle function}$ Bandlimited $\Delta f = \text{half BW.}$</td>
<td></td>
</tr>
<tr>
<td>$x_p(t) = x(t) * s(t)$</td>
<td>$\tilde{X}(f)\tilde{S}(f)$ Periodic in time</td>
</tr>
<tr>
<td>$x_p(t) = \sum_k a_k e^{2\pi ikt/\Delta}$</td>
<td>$\Delta^{-1} \sum_k \tilde{X}(k/\Delta)\delta(f - k/\Delta)$ Fourier series</td>
</tr>
</tbody>
</table>

Also:

Uncertainty principle

\[ \text{Also:} \]

Uncertainty principle
Utility of FT Theorems

• Conceptual, making guesses, actually obtaining FTs of complicated functions

• Basis for algorithms:
  – E.g. Shift theorem:
    \[ x(t - t_0) \leftrightarrow \hat{X}(f)e^{-2\pi ift_0} \]

• Can use matched filtering in the t domain or estimate the phase slope of the Fourier transform.

• There is a theoretical limit to how precisely an object can be located \( \sim 1 / (\text{signal to noise ratio}) \)

• In pulse localization, frequency domain approaches are easier to implement and achieve the theoretical limit
Hi, Dr. Elizabeth?
Yeah, uh... I accidentally took the Fourier transform of my cat...

Meow!
Discrete Fourier Transform (DFT)

The DFT of a uniformly spaced array of data \( \{x_n, n = 0, \ldots, N - 1\} \) is defined as

\[
\tilde{X}_k = N^{-1} \sum_{n=0}^{N-1} x_n e^{-2\pi i n k / N}
\]

The inverse transform is

\[
x_n = \sum_{k=0}^{N-1} \tilde{X}_k e^{+2\pi i n k / N}
\]

which may be shown to have the correct normalization, etc. by substituting for \( \tilde{X}_k \):

\[
x_n = C \sum_{k=0}^{N-1} \tilde{X}_k e^{+2\pi i n k / N}
\]

\[
= C \sum_{k=0}^{N-1} N^{-1} \sum_{n'=0}^{N-1} x_{n'} e^{-2\pi i n' k / N} e^{+2\pi i n k / N}
\]

\[
= C N^{-1} \sum_{n'=0}^{N-1} x_{n'} \sum_{k=0}^{N-1} e^{2\pi i (n-n')k / N}
\]

\[
= C N^{-1} \sum_{n'=0}^{N-1} x_{n'} N \delta_{nn'}
\]

\[
= C x_n = x_n \quad \text{for } C \equiv 1
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\[
= C x_n = x_n \quad \text{for } C \equiv 1
\]

The FFT is simply a fast algorithm for calculating the DFT.

It exploits redundancies in the exponential when N is factorable; esp. \( N = 2^M \) but any prime will do.

A direct calculation of the DFT requires \( \sim N^2 \) operations.

The FFT requires \( \sim N \log N \) operations.
• The normalization calculation expresses the orthogonality property of the basis functions (exponentials).

• In calculations of this kind, identifying the implied δ-function is typical. Here it relied on summing over products of basis functions.

• In other contexts, the δ-function will arise from statistical independence of random variables.

• What happens if the sampling of \( x_n \) is not uniform? Orthogonality is broken. So what?

• Notation: we often designate that \( x_n \) and \( \tilde{X}_k \) are Fourier transform pairs by writing

\[
x_n \leftrightarrow \tilde{X}_k
\]

and we say that \( n \) and \( k \) are conjugate variables.

• \( n \) can be time, a spatial coordinate, a wavelength, anything.

• Extension to \( N_D \) dimensions is trivial:
  – E.g. a 2D DFT of an \( N \times M \) size object can be calculated as a series of \( M \) 1D-DFTs of length \( N \) followed by \( N \) 1D-DFTs of length \( M \)

• From a systems point of view, the DFT is a linear operation and does not lose information.

• An alternative approach is to fit a sinusoidal model to the data using an assumed frequency \( k/N \). It can be shown that the DFT is the least-squares solution for the amplitudes of the sinusoids for all \( k \). We will show this later on by using matrix algebra to solve for least-squares solutions.
Fourier Comments

• Fourier series/transforms involve exponential basis functions that are orthogonal over a relevant interval (e.g. [0, N-1] for the DFT)

• If there are unknown values of the function in either domain, then orthonormality is broken
  • gaps
  • nonuniform sampling
  • misestimated values before FT

• The three FT forms are similar but not always interchangeable
  • sampling and transforming do not commute

• Issues:
  • aliasing
  • periodicity and convolution
Vector form of DFT

A time series can be written as a data vector.
So can its Fourier transform.
An N-point FT can be written as the product of a matrix and the data vector.

What does this matrix look like?
Symmetry Properties of the DFT

\[ \hat{X}_k = N^{-1} \sum_{n=0}^{N-1} x_n e^{-2\pi i nk/N} \quad \text{and} \quad x_n = \sum_{k=0}^{N-1} \hat{X}_k e^{2\pi i nk/N} \]

• Periodic with period \( N \) in both domains

• Time series:
  – Discrete functions with sample intervals \( \delta t \) and \( \delta f \)
  – \( T = N\delta t \) = time total time span
  – \( \delta f = 1/T \)
  – Nyquist frequency = maximum frequency that is represented without distortion:
    \[ f_N = \frac{N\delta f}{2} = \frac{1}{2\delta t} \]
Symmetry Properties of the DFT

\[ \tilde{X}_k = N^{-1} \sum_{n=0}^{N-1} x_n e^{-2\pi i nk/N} \quad \text{and} \quad x_n = \sum_{k=0}^{N-1} \tilde{X}_k e^{2\pi i nk/N} \]

- Hermitian (show by substituting into the DFT expression for \( x_n \))

\[ x_n^* \iff \tilde{X}_{N-k}^* \]

If \( x_n \) is real then

\[ x_n \iff \tilde{X}_{N-k}^* \]

\[ \tilde{X}_{N-k}^* = \tilde{X}_k \]

- The symmetry properties tell us how to fill an array with data to achieve specific results
- What are the symmetry properties of a 2D DFT?

\[ \tilde{X}_{kl} = \frac{1}{NM} \sum_n \sum_m x_{nm} e^{-2\pi i (nk+ml)/NM} \]
Mapping Frequency to Frequency Bin

Consider

\[ x(t) = e^{i\omega_0 t} = e^{2\pi i f_0 t} \]

In an N-point DFT, in which bin does the signal fall (mostly)?

As before we have \( T = N\delta t \) and \( \delta f = 1/T \)

The frequency mapping is

\[ f_j = j\delta f \quad \text{for } j = 0, \ldots, N/2 \]
\[ = (N - j)\delta f \quad \text{for } j = N/2 + 1, \ldots, N - 1 \]

The Nyquist frequency is the maximum frequency in the spectrum

\[ f_N = \frac{1}{2\delta t} \]

So for \( f_0 \leq f_N \) we the corresponding frequency bin \( k \) in the DFT is

\[ k_0 = f_0 N\delta t \]

Negative frequencies correspond to \( k \) above the Nyquist frequency.

Frequencies \( |f_0| > f_N \) are still represented in the DFT (remember ... it is lossless) but they appear at aliased frequencies.

The frequency bin \( k_0 \): varies with \( N \) for fixed \( f_0 \) and \( \delta t \) and varies with \( \delta t \) for fixed \( f_0 \) and \( N \).
How do we fill an array to get a real signal in the other domain?

• $k = 0, 1, \ldots, N-1$
• Need to know $N/2-1+2 = N/2+1$ unique values
A guided tour of the fast Fourier transform

The fast Fourier transform algorithm can reduce the time involved in finding a discrete Fourier transform from several minutes to less than a second, and also can lower the cost from several dollars to several cents

G. D. Bergland    Bell Telephone Laboratories, Inc.

For some time the Fourier transform has served as a bridge between the time domain and the frequency domain. It is now possible to go back and forth between waveform and spectrum with enough speed and economy to create a whole new range of applications for this classic mathematical device. This article is intended as a primer on the fast Fourier transform, which has revolutionized the digital processing of waveforms. The reader's attention is especially directed to the IEEE Transactions on Audio and Electroacoustics for June 1969, a special issue devoted to the fast Fourier transform.

crete version of the Fourier transform (DFT) that must be understood and used. Although most of the properties of the continuous Fourier transform (CFT) are retained, several differences result from the constraint that the DFT must operate on sampled waveforms defined over finite intervals.

The fast Fourier transform (FFT) is simply an efficient method for computing the DFT. The FFT can be used in place of the continuous Fourier transform only to the extent that the DFT could before, but with a substantial reduction in computer time. Since most of the problems associated with the use of the fast Fourier transform actually stem from an incomplete or incorrect under-
Cyclical vs Non-cyclical Convolution & Correlation

From Bergland, "A Guided Tour of the FFT", IEEE Spectrum, 1969, 6 (Bell Labs)
Noncyclical Procedures

FIGURE 16. Noncyclical convolution of two finite signals analogous to that performed by the FFT algorithm.

FIGURE 17. A method for convolving a finite impulse response with an infinite time function by performing a series of fast Fourier transforms.