1. **(Motion of the Sun, 12’)** The Sun is currently located 30 pc north of the Galactic midplane and moving away from it with a velocity \( v_z = 7 \text{ km s}^{-1} \). Since the gravitational force on the Sun due to other disk matter is directed toward the midplane, the Sun exhibits oscillatory vertical motion about the Galactic midplane.

   (a) Derive an expression for the gravitational acceleration vector at a height \( z \) above the midplane, assuming that the Sun always remains inside the disk of constant density \( \rho \). (Note that the disk of the Milky Way has a radius that is much larger than its thickness. So you can treat the disk as being infinite in the radial direction).

   (b) Show that the motion of the Sun in the \( z \)-direction can be described by the equation
   \[
   \frac{d^2z}{dt^2} + kz = 0.
   \]
   Express \( k \) in terms of \( \rho \) and \( G \). If the total mass density in the solar neighborhood is \( \rho \approx 0.1 \text{ } M_\odot \text{ pc}^{-3} \), estimate the oscillation period (in years).

   (c) Estimate the amplitude of the solar oscillation (i.e., the maximum height it can reach above the midplane).

   (d) How many vertical oscillations does the Sun execute during one orbital period around the Galactic center?

2. **(Drag Force on Halo Stars, 10’)**

   The oldest (Pop II) stars are thought to have formed while our Galaxy was still collapsing towards its current disk-like shape.

   (a) Estimate the typical velocity of such a star at a radius of 10 kpc from the Galactic center. Assume a Galactic mass of \( 2 \times 10^{11} M_\odot \) interior to 10 kpc. **Hint:** You may want to look at page 13 of Schneider to learn about Virial theorem. Try to understand it yourself or ask me.

   (b) Assume that, when the collapse of the Galaxy is complete, the remaining gas is in a disk 200 pc thick with an average density of 1 hydrogen atom per cm\(^3\). As the halo star of (a) passes through the disk, it will experience a drag force from the gas in the disk. Make a crude estimate of the fraction of the star’s vertical momentum lost on each passage through the disk. **Hint:** The drag force on a sphere moving at speed \( V \) through a medium of density \( \rho \) is given by \( F = -C_D \pi R^2 \rho V^2 \), with \( C_D \approx 1 \).

   (c) Estimate how long it will take for the star to lose most of its initial vertical momentum and be absorbed in the disk.

3. **(Differential Galactic Rotation Near the Sun, 10’)**

   As discussed in class, using Doppler effect, one can measure the radial velocity \( u_\parallel \) of a star relative to the Sun. For nearby stars, one can also measure the star’s transverse velocity \( u_\perp \) relative to the Sun (this involves measuring the star’s changing position in the plane of the sky and measuring its distance \( D \)). This problem will examine how, by measuring \( u_\parallel \) and \( u_\perp \) for nearby stars, one can determine the Galaxy’s rotation curve in the solar neighborhood.

   Consider the Galaxy with circular rotation curve \( V_c(R) \). Look at Fig.2.13 of Schneider for the definitions of the Galactic coordinates. Neglect peculiar velocities since they are quite small (i.e., assume all stars move in a circular orbit). For a star at distance \( D \), longitude \( l \) and latitude \( b = 0 \), derive the expressions for \( u_\parallel \) and \( u_\perp \). Your answers should depend on \( D \), \( l \), \( R_0 \) (the distance of the Sun from the Galactic center) and the function \( V_c(R) \). What can we learn about the function \( V_c(R) \) if we measure \( u_\parallel \) and \( u_\perp \) for stars at different \( D \) and \( l \)?

4. **(Gravitational Lensing, 13’)** A galaxy similar to the Milky Way (total mass \( \sim 10^{12} M_\odot \)) lies half way between a distant quasar (at 1 Gpc) and the Earth.
(a) Estimate the angular separation (in arcsecond) of the quasar images. (You may treat the galaxy as a point mass; i.e., the whole galaxy acts as a lens. For our purpose here, a quasar is simply bright point light source).

(b) Estimate the typical image separation (in kpc) on the lens plane. Do you expect the point-mass approximation to be a good one?

(c) Consider microlensing of the quasar by individual objects (stars, MACHO’s) in the galaxy. Show that the optical depth \( \tau \) can be written in terms of the column density \( \Sigma \) (in g cm\(^{-2}\)) of mass through the galaxy along the line of sight:

\[
\tau = \frac{\Sigma}{\Sigma_c},
\]

where \( \Sigma_c \) is the critical column density:

\[
\Sigma_c = \frac{c^2}{4\pi GD} = 0.35 \left( \frac{1 \text{ Gpc}}{D} \right) \text{ g cm}^{-2}, \quad D = \frac{D_d(D_s - D_d)}{D_s},
\]

(The symbols have their usual meaning).

(d) Assume that the halo of the galaxy is a sphere with radius \( R = 50 \) kpc and with uniform mass density (You may ignore the disk of the galaxy). What is the probability that the quasar is microlensed by objects in the intervening galaxy?