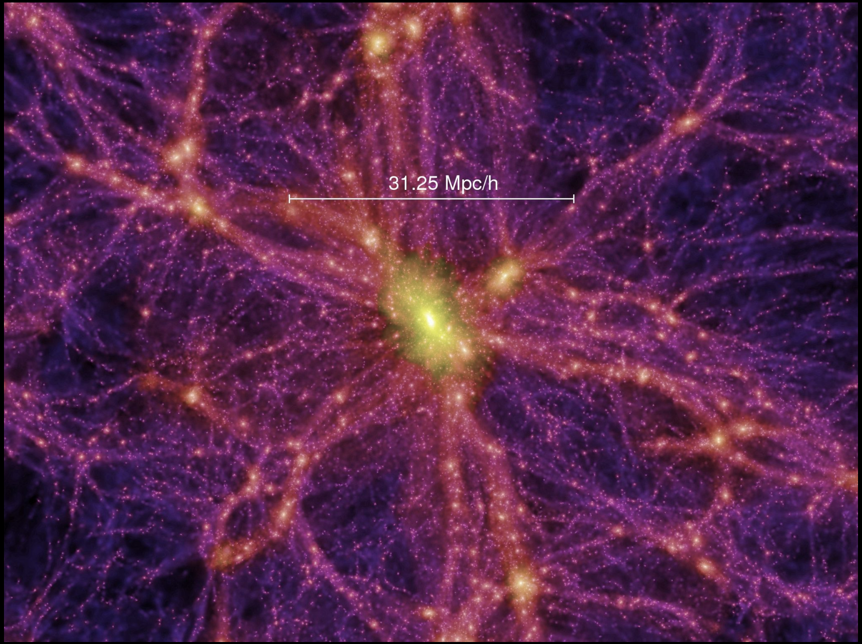
The Final Parsec Problem: How Supermassive Black Holes Merge

Ross Jennings • December 5, 2018

Structure Formation and Supermassive BHs



- Idea: Every galaxy has a central BH.
- Structures merge hierarchically.
- As galaxies merge, so do their BHs?

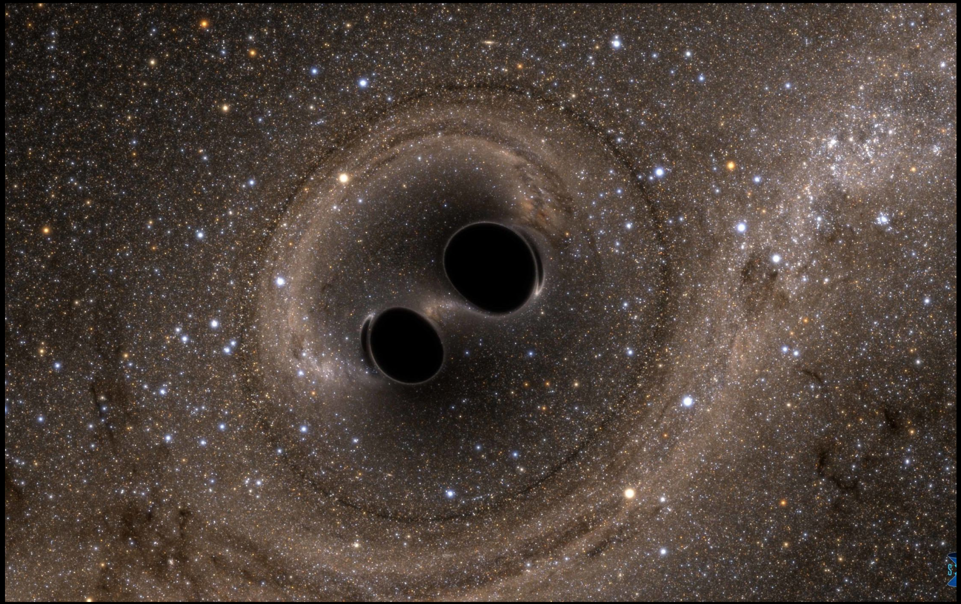
$$f_{\mathrm{GW}} = \frac{2\Omega_K}{2\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{a^3}}$$

$$f_{\mathrm{GW}} \approx (2 \times 10^{-8}\ \mathrm{Hz})\left(\frac{M}{10^6\, M_\odot}\right)^{\frac{1}{2}}\left(\frac{a}{10^{-3}\,\mathrm{pc}}\right)^{-\frac{3}{2}}$$

$$f_{\mathrm{merg}} \sim 3 \times 10^{-2}\ \mathrm{Hz}\left(\frac{M}{10^6\, M_\odot}\right)^{-1}$$

GW strain estimates

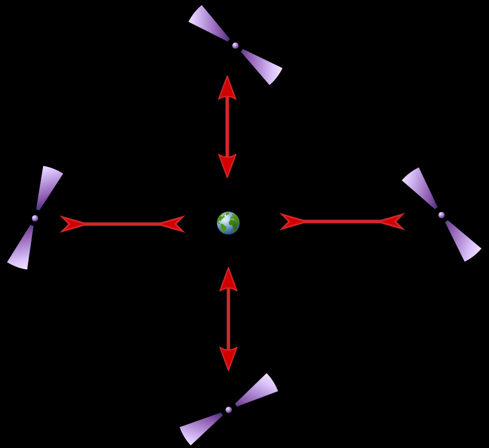
$$\bar{h}_{ij} = \frac{2G}{rc^4}\ddot{I}_{ij}$$

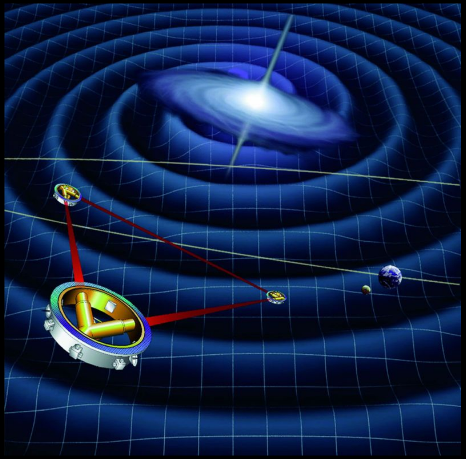


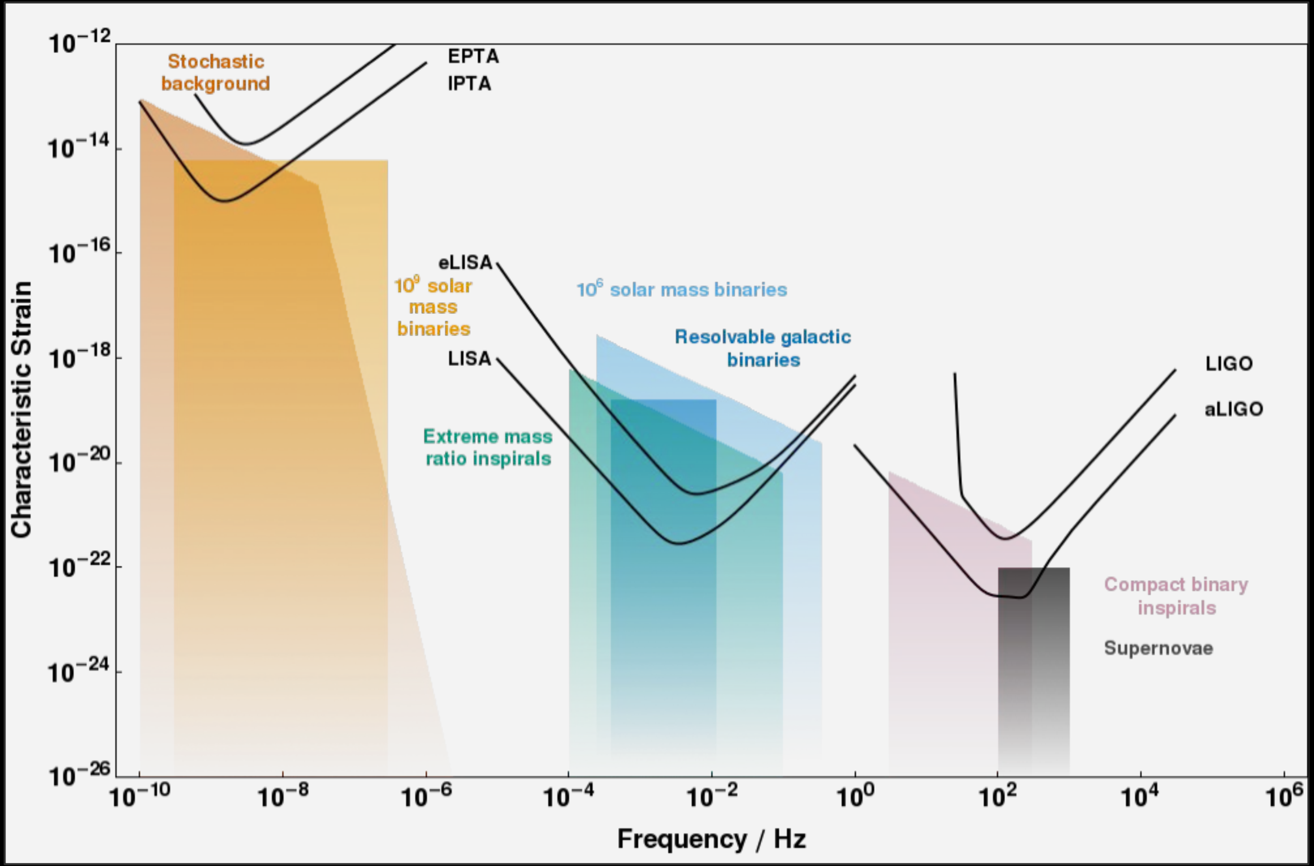
$$h \sim \frac{G}{rc^4}\Omega^2 \mu a^2 \sim \frac{G^2 M \mu}{arc^4}$$

$$h \sim 2 \times 10^{-18}\left(\frac{M}{10^6\, M_\odot}\right)^2 \left(\frac{a}{10^{-3}\,\mathrm{pc}}\right)^{-1} \left(\frac{r}{1\ \mathrm{Mpc}}\right)^{-1} \frac{m}{M}$$

Detectability: PTAs, LISA





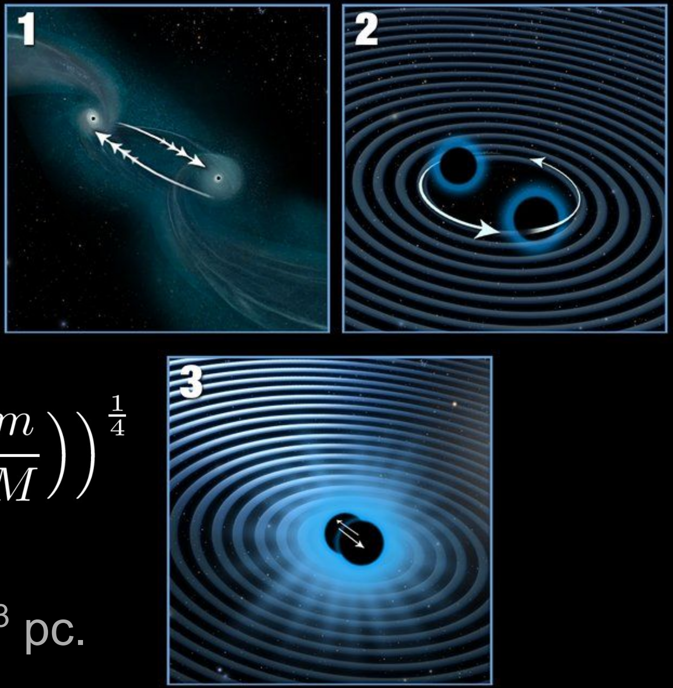


Merging by GW emission takes a long time!

Evolution of *a* from GW luminosity:

$$\frac{d}{dt}\left(-\frac{GM\mu}{2a}\right) = -\frac{32G^4M^3\mu^2}{5c^5a^5}$$

$$a = (1.1 \times 10^{-3}\ \mathrm{pc})\left(\frac{M}{10^6\,M_\odot}\right)^{\frac{3}{4}}\left(\frac{T}{14\ \mathrm{Gyr}}\right)^{\frac{1}{4}}\left(\frac{m}{M}\left(1+\frac{m}{M}\right)\right)^{\frac{1}{4}}$$



To merge within a Hubble time, must start within ~10^{-3} pc.

How do they get this close?

Getting closer: dynamical friction

Chandrasekhar:

$$\vec{F} \sim -C \frac{G^2 M^2 \rho}{v^2} \hat{v}$$

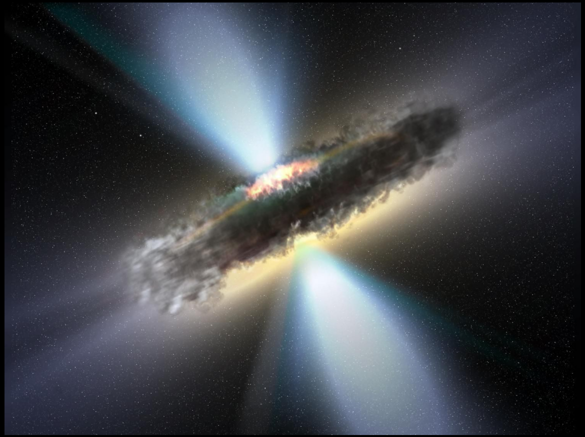
"Hardening radius":

$$-\frac{G\mu}{2a_h} \sim \langle v^2 \rangle$$

$$a_h \approx (0.22 \text{ pc}) \left(\frac{m}{10^6 M_\odot} \right) \left(\frac{v_{\rm rms}}{100 \text{ km/s}} \right)^{-1} \left(1 + \frac{m}{M} \right)^{-1}$$

Can they get close enough?

$$\frac{a_h}{a_{\rm GW}} \sim 200 \left(\frac{M}{10^6\, M_\odot}\right)^{-\frac{1}{4}} \left(\frac{v_{\rm rms}}{100\ {\rm km/s}}\right)^{-1} \left(\frac{T}{14\ {\rm Gyr}}\right)^{\frac{1}{4}}.$$



Hence “final parsec problem”.

Ways to get around it:

- Galaxies aren’t really spheres.
- Stalled binaries will eventually meet a third BH.

A possible solution: triaxiality

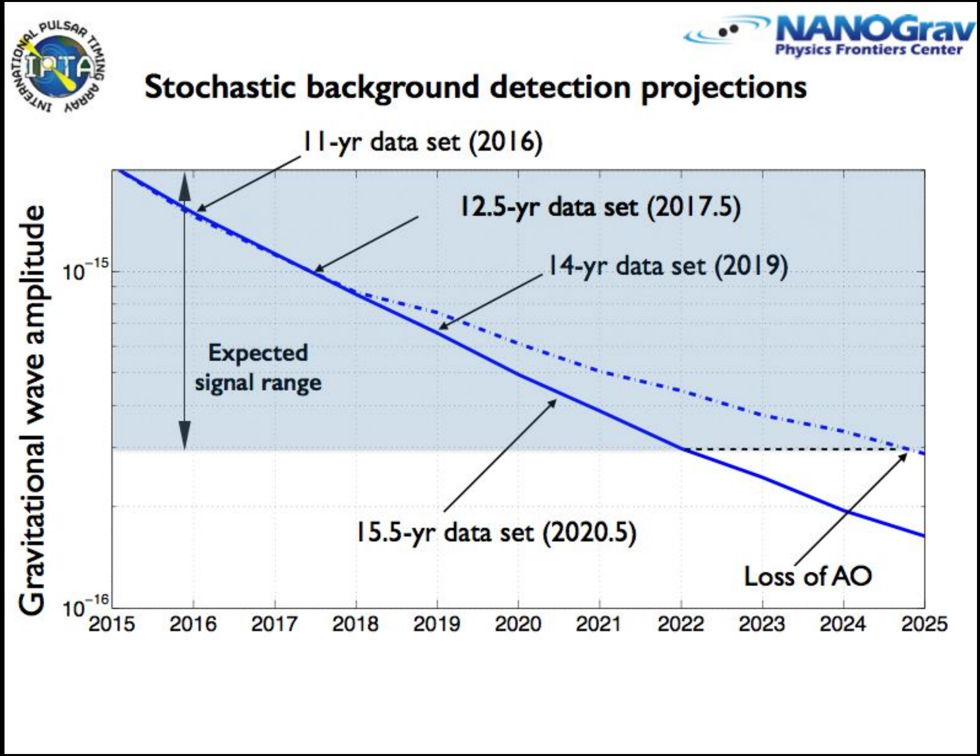
Triaxiality → Loss cone refilling

- “Loss cone”: phase space region of stars with ang. mom. small enough to interact with (& be ejected by) the central BH
- In a spherical galaxy, ang. mom. is conserved, so the loss cone is eventually emptied, & binary stalls.
- In realistic galaxies (triaxial ellipsoids), global torques can refill the loss cone.

The “worst” case: multi-BH interactions

If all SMBH binaries stall:

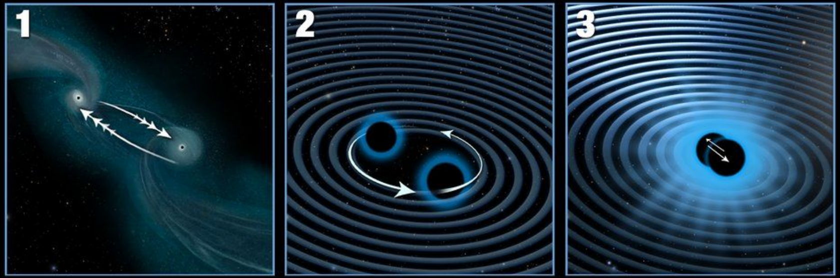
- Cosmic population of stalled binaries exists
- Eventually, another galaxy will merge & add its SMBH
- Three-body interactions between SMBHs can shrink orbit into GW regime.



References

- Ryu, T., Perna, R., Haiman, Z., et al. (2018), “Interactions between multiple supermassive black holes in galactic nuclei: a solution to the final parsec problem”, MNRAS 473, pp. 3410–3433.
- Gualandris, A., Read, J., Dehnen, W., & Bortolas, E. (2017), “Collisionless loss cone refilling: there is no final parsec problem”, MNRAS 464, pp. 2301–2310.
- Milosavljević, M., & Merritt, D. (2003), “The Final Parsec Problem”, in *The Astrophysics of Gravitational Wave Sources*, AIP Conference Series, v. 686, pp. 201–210.

Questions?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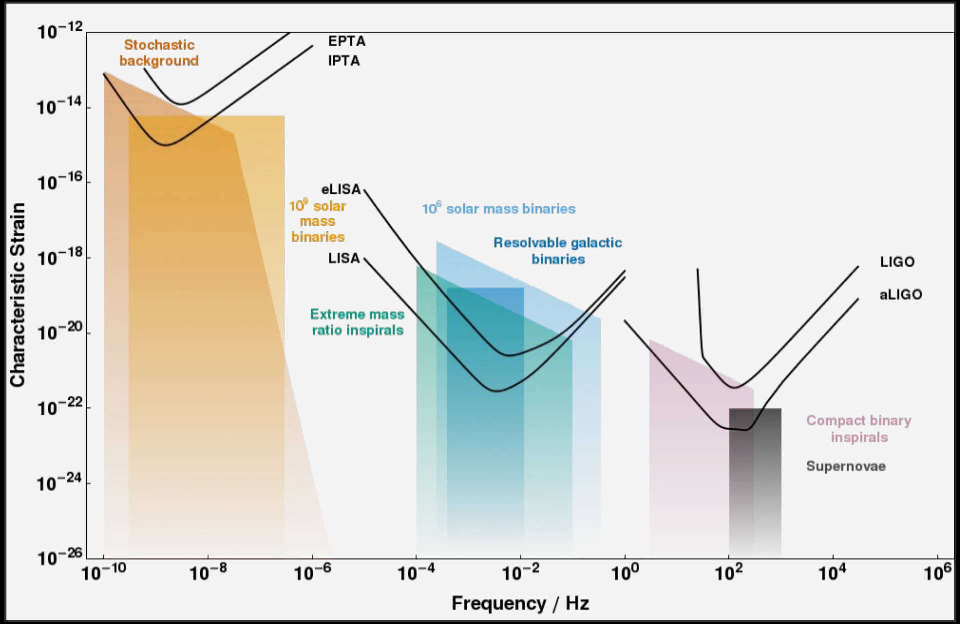
Characteristic Strain
Frequency / Hz
10^{-12}
10^{-14}
10^{-16}
10^{-18}
10^{-20}
10^{-22}
10^{-24}
10^{-26}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
10^{6}
Stochastic background
EPTA
IPTA
eLISA
LISA
10^9 solar mass binaries
10^6 solar mass binaries
Resolvable galactic binaries
Extreme mass ratio inspirals
LIGO
aLIGO
Compact binary inspirals
Supernovae

Mergers of supermassive black holes and the final parsec problem

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Modern cosmology has established a model in which galaxies and galaxy clusters form by repeated mergers over the lifetime of the universe, and where nearly every galaxy contains a central supermassive black hole with a mass of at least $10^6\,M_\odot$, with some being as large as a few times $10^9\,M_\odot$. It is reasonable to assume that the black holes at the centers of merging galaxies will themselves eventually merge, particularly because dynamical friction will draw them toward the center of the merged galaxy. Before they completely merge, the supermassive black holes will form a binary system, which is in principle observable through the gravitational waves it emits, and potentially through modifications to the electromagnetic waves emitted as the black holes accrete matter from the surrounding galaxy. Although a few promising candidates have been identified, no conclusive evidence of these supermassive black hole binaries yet exists. However, their gravitational wave signatures may soon be detected by pulsar timing arrays and/or space-based interferometers such as LISA. In the following, I will derive estimates for the gravitational wave frequencies and amplitudes that should be expected from such systems, and discuss the mechanisms responsible for shrinking their orbits to the point where they can merge via gravitational wave emission. Simple arguments suggest that the binaries may not be able to shrink rapidly enough via interactions with the stars in the center of the merged galaxy to reach the point where they can merge via gravitational wave emission within the current age of the universe. This is known as the "final parsec problem", and I will discuss both the source of the problem and some mechanisms which may resolve it.

1 Gravitational wave parameters

Here I derive estimates and scaling relations for the frequency and amplitude of the gravitational waves produced by a supermassive black hole binary.

1.1 Frequency

The frequency of gravitational waves emitted by a binary system is twice the orbital frequency. The orbital angular frequency is well approximated by the Keplerian value

$$\Omega_K = \sqrt{\frac{GM}{a^3}}. \tag{1}$$

Here a is the binary separation (semimajor axis), and $M = M_1 + M_2$ is the total mass. It follows that the gravitational wave frequency is

$$f_{\mathrm{GW}} = \frac{2\,\Omega_K}{2\pi} = \frac{1}{\pi}\sqrt{\frac{GM}{a^3}}. \tag{2}$$

Using nominal values for M and a of $10^6\,M_\odot$ and $10^{-3}\,\mathrm{pc}$, respectively, gives

$$f_{\mathrm{GW}} = 2.1 \times 10^{-8}\ \mathrm{Hz}\left(\frac{M}{10^6\,M_\odot}\right)\left(\frac{a}{10^{-3}\,\mathrm{pc}}\right)^{-\frac{3}{2}}. \tag{3}$$

The relevance of the nominal separation will be seen later; it is approximately the maximum separation that two supermassive black holes can have to merge within the age of the universe through gravitational wave emission alone.

A supermassive black hole binary will emit the highest-frequency gravitational radiation when it is at the point of merger. We can obtain a rough estimate of the frequency at merger by setting the separation a in the above equations equal to the radius of the innermost stable circular orbit (ISCO):

$$a = r_{\mathrm{ISCO}} = \frac{6GM}{c^2} = 2.9 \times 10^{-7}\ \mathrm{pc}\left(\frac{M}{10^6\,M_\odot}\right). \tag{4}$$

While strictly speaking the ISCO is only well-defined for a test mass of negligible size orbiting a single black hole or relativistic neutron star, it gives a reasonable estimate of the separation below which a pair of black holes will merge. Substituting $a = 6GM$ into (2) gives

$$f_{\rm merg} = \sqrt{\frac{GM}{(6GM)^3}} = \frac{1}{6\sqrt{6}GM} = 4.4 \times 10^{-3}\ \text{Hz}\left(\frac{M}{10^6\, M_\odot}\right)^{-1}. \tag{5}$$

Of the detectors sensitive to gravitational waves from supermassive black hole binaries, pulsar timing arrays are most sensitive at frequencies between about 10^{-9} and 10^{-7} Hz, and space-based interferometers such as LISA are most sensitive at frequencies between 10^{-4} and 10^{-2} Hz. From the scaling relations above, it is clear that $10^6\, M_\odot$ binaries merge in the LISA band, and binaries of $10^9\, M_\odot$ and higher emit gravitational waves in the nanohertz band (detectable by PTAs) when their separations are of order 10^{-5} pc, a good deal smaller than the nominal radius used above.

1.2 Strain

The gravitational wave strain produced by a binary system is given by

$$h_{ij} = \frac{2G}{rc^4}\ddot{I\!\!\!-}_{ij}, \tag{6}$$

where $I\!\!\!-$ is the trace-reversed moment of inertia tensor. The moment of inertia I scales as μa^2, where μ is the reduced mass of the binary and a is the separation. It follows that the gravitational wave strain scales as

$$h \sim \frac{G}{rc^4}\Omega^2 \mu a^2, \tag{7}$$

since each derivative introduces a factor of the orbital frequency Ω. In more detail, the amplitude is directionally dependent, but this estimate is sufficient to give the correct order of magnitude. Using the Keplerian orbital frequency (1) for Ω gives

$$h \sim \frac{G^2 M\mu}{arc^4}. \tag{8}$$

Using nominal values of 10^{-3} pc for a, 1 Mpc for r, and $10^6\, M_\odot$ for the mass M_1 of the larger black hole, this becomes

$$h \sim 2.3 \times 10^{-18}\left(\frac{M_1}{10^6\, M_\odot}\right)^2\left(\frac{a}{10^{-3}\ \text{pc}}\right)^{-1}\left(\frac{r}{1\ \text{Mpc}}\right)^{-1}\frac{M_2}{M_1}. \tag{9}$$

1.3 Lifetime

To find the separation $a_{\rm GW}$ from which a supermassive black hole binary can merge within the lifetime of the universe, we can set the time derivative of the binding energy equal to the negative gravitational wave luminosity $L_{\rm GW}$. Because the intention here is only to provide a rough estimate of this distance, I will use the Newtonian form of the binding energy. This gives

$$\frac{d}{dt}\left(-\frac{GM\mu}{2a}\right) = \frac{GM\mu}{2a^2}\frac{da}{dt} = -\frac{32\,G^4M^3\mu^2}{5\,c^5a^5}. \tag{10}$$

Separating the variables, we have

$$a^3\,da = -\frac{64\,G^3M^2\mu}{5\,c^5}\,dt \;\Rightarrow\; \frac{1}{4}a^4 - \frac{1}{4}a_0^4 = -\frac{64\,G^3M^2\mu}{5\,c^5}(t-t_0), \tag{11}$$

where a_0 is the binary separation at the arbitrary reference time t_0. If we take t_0 to be the time of merger, then $a_0 = 0$, and, defining $\tau = t_0 - t$ to be the time until merger, we have

$$a_{\rm GW}^4 = \frac{256\,G^3M^2\mu}{5\,c^5}\,\tau. \tag{12}$$

Plugging in numbers produces the scaling relation

$$a_{\rm GW} = 2.2\times10^{-3}\ {\rm pc}\left(\frac{M_1}{10^6\,M_\odot}\right)^{\frac{3}{4}}\left(\frac{\tau}{14\ {\rm Gyr}}\right)^{\frac{1}{4}}\left(\frac{M_2}{M_1}\left(1+\frac{M_2}{M_1}\right)\right)^{\frac{1}{4}}, \tag{13}$$

where M_1 is the mass of the larger black hole and M_2 is the mass of the smaller one. This is the source of the nominal value of 10^{-3} pc for the binary separation used previously.

2 The final parsec problem

The scale of the smallest separation attainable by supermassive black hole binaries through dynamical friction and scattering of stars is set by the point where the binding energy per unit mass of the binary is comparable to the velocity dispersion of the stars in the center of the merged galaxy. Symbolically, this is

$$\frac{G\mu}{2a_h} = \langle v^2\rangle, \tag{14}$$

where a_h is the "hardening radius" at which dynamical friction ceases to be effective at reducing the separation of the binary, and $\langle v^2 \rangle$ denotes the mean squared velocity of stars in the center of the galaxy containing the binary. Solving for the hardening radius a_h gives

$$a_h = \frac{G\mu}{2\langle v^2 \rangle} \approx 0.22\ \mathrm{pc} \left(\frac{M_2}{10^6\, M_\odot} \right) \left(\frac{v_{\mathrm{rms}}}{100\ \mathrm{km/s}} \right)^{-1} \left(1 + \frac{M_2}{M_1} \right)^{-1}, \tag{15}$$

where again M_1 is the mass of the larger black hole and M_2 is the mass of the smaller one. This is clearly significantly larger than the separation a_{GW} necessary for the black holes to merge within the age of the universe, as computed earlier. This observation is the basis of the "final parsec problem", which was first described by Milosavljević and Merritt (2003).

2.1 Potential solutions

The hardening limit arises because stars in the galaxy must have sufficiently small angular momentum to interact with the binary. The phase space region containing stars with sufficiently small angular momenta is called the "loss cone", and, in a spherical galaxy, it is eventually emptied out as stars with sufficiently small angular momenta are ejected from the galaxy through interactions with the central black hole binary.

However, in more realistic galaxies, a significant degree of triaxiality is present, so the angular momentum of individual stars is not conserved – in other words, the gravitational potential of the galaxy creates global torques. Gualandris et al. (2017) explore these global torques as a possible solution to the final parsec problem, and find that with realistic levels of triaxiality, the loss cone refills sufficiently fast that supermassive black hole binaries can reach a_{GW} reasonably quickly.

Another possibility is that the stalling described by the final parsec problem really happens, and that there exists a cosmic population of stalled supermassive black hole binaries. In this case, when a galaxy containing a binary merges with another galaxy, a three-body system will be created at the center of the new, merged galaxy, and chaotic three-body dynamics will ensue, with the most likely endpoint being that one of the black holes is ejected and the other two become even closer together than the original pair, perhaps even close enough to merge by gravitational wave emission. This possibility is explored by Ryu et al. (2018), who find that in such a scenario, significant levels of low-frequency gravitational wave emission are still expected, albeit at a lower level than would be predicted if all supermassive black hole binaries efficiently merged.

References

Ryu, T., Perna, R., Haiman, Z., et al. (2018), "Interactions between multiple supermassive black holes in galactic nuclei: a solution to the final parsec problem", MNRAS 473, pp. 3410–3433.

Gualandris, A., Read, J., Dehnen, W., & Bortolas, E. (2017), "Collisionless loss cone refilling: there is no final parsec problem", MNRAS 464, pp. 2301–2310.

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