

Physics 6553 : Problem Set 1

Due Thursday , Sept 6, 2012

Reading: Chapter 1 and sections 2.3 and 2.4 of Wald. You may also find it useful to consult sections 1.4 to 1.7 of Carroll, sections 2.2 to 2.7, 2.9, 3.2, 3.3 and 3.5 of Misner, Thorne and Wheeler (henceforth MTW), or sections 20.1 to 20.3 of Hartle.

1. [5 points] *Double Dual of a Vector Space:* Given a finite dimensional vector space V , we defined a map from V to V^{**} taking \vec{v} to $F_{\vec{v}}$ where

$$F_{\vec{v}}(\mathbf{w}) = \mathbf{w}(\vec{v})$$

for any dual vector \mathbf{w} in V^* . Show that this mapping is linear and bijective, so that V and V^{**} are naturally isomorphic. Did you need to use the fact that V is finite dimensional?

2. [10 points] *Properties of Tensor Product:* Given finite dimensional vector spaces U , V , and W , establish the followign natural isomorphisms:

$$(U \otimes V)^* \simeq U^* \otimes V^*$$
$$(U \otimes V) \otimes W \simeq U \otimes (V \otimes W).$$

In the second case, show that both of these spaces are naturally isomorphic to the set of multilinear maps from $U^* \times V^* \times W^*$ to the real numbers.

3. [5 points] *Reinterpretation of tensors:* Show that the set of multilinear maps

$$T : V \times \dots \times V \times V^* \times \dots \times V^* \rightarrow V \otimes \dots \otimes V \otimes V^* \otimes \dots \otimes V^*$$

is naturally isomorphic to the tensor space $T_{s+s'}^{r+r'}(V)$. Here on the left hand side there are s copies of V and r copies of V^* , and on the right hand side there are r' copies of V and s' copies of V^* .

4. [5 points] *Outer products:* Suppose that A is a tensor in $T_s^r(V)$ and B is a tensor in $T_{s'}^{r'}(V)$. Using the natural isomorphism $T_s^r(V) \otimes T_{s'}^{r'}(V) \simeq T_{s+s'}^{r+r'}(V)$, show that the tensor product of A and B is given by the formula

$$(A \otimes B)(\mathbf{w}^1, \dots, \mathbf{w}^{r+r'}, \vec{v}_1, \dots, \vec{v}_{s+s'}) = A(\mathbf{w}^1, \dots, \mathbf{w}^r, \vec{v}_1, \dots, \vec{v}_s) \\ \times B(\mathbf{w}^{r+1}, \dots, \mathbf{w}^{r+r'}, \vec{v}_{s+1}, \dots, \vec{v}_{s+s'}).$$

5. [5 points] *Quadratic Forms:* Suppose that q is a nondegenerate, symmetric tensor in $T_2^0(V)$. Show that there exists a choice of basis \vec{e}_α of V for which the matrix of components $q_{\alpha\beta}$ is diagonal with all the diagonal components either +1 or -1. [Hint: one method is to use induction in the dimension of V .]