Physics 6553 : Problem Set 11

Due Thursday, Nov 15, 2012

1. *Near-horizon approximation to Schwarzschild spacetime:* [5 points] In this problem we will show that the Rindler metric is a good approximation to the Schwarzschild metric near to the horizon of the black hole. Start from the Schwarzschild metric in the form

$$ds^2 = -w(r)dt^2 + \frac{1}{w(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $w(r) = 1 - 2M/r$. Focus attention on the point $\mathcal{P}$ given by $r = 2M$, $t = \theta = \phi = 0$. Find a transformation to a new set of coordinates $\bar{t}, \bar{x}, \bar{y}, \bar{x}$ in a neighborhood of $\mathcal{P}$ so that $\bar{x} = \bar{y} = \bar{z} = \bar{t} = 0$ at $\mathcal{P}$, and so that the metric takes the approximate Rindler form

$$ds^2 = \left[-\bar{x}^2 d\bar{t}^2 + d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2\right]\left[1 + O\left(\frac{\bar{x}^2}{M^2}, \frac{\bar{y}^2}{M^2}, \frac{\bar{z}^2}{M^2}\right)\right].$$

2. *Action principle for a point particle interacting with gravity and with EM fields:* [10 points] The combined action for a point particle, EM fields and gravitational fields is given by specifying a functional S of a spacetime metric $g_{\alpha\beta}(x^\mu)$, a 4-potential $A_\alpha(x^\mu)$ and a worldline $x^\alpha(\tau)$. Consider the action

$$S = S[g_{\alpha\beta}(x^\mu), A_\alpha(\mu), x^\alpha(\tau)] = S_{\text{gravity}} + S_{\text{matter}}, \tag{1}$$

where

$$S_{\text{gravity}} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R, \tag{2}$$

and

$$S_{\text{matter}} = -\frac{1}{16\pi}\int d^4x\sqrt{-g}\,F_{\alpha\beta}F^{\alpha\beta} + \int d\tau\,\left[e\dot{x}^\alpha(\tau)A_\alpha[x^\mu(\tau)] - m\sqrt{-g_{\alpha\beta}[x^\gamma(\tau)]\dot{x}^\alpha\dot{x}^\beta}\right], \tag{3}$$

and where $F_{\alpha\beta} \equiv 2\nabla_{[\alpha}A_{\beta]}$, m is the particle's mass, e is the particle's charge, and $\dot{x}^\mu = dx^\mu/d\tau$.

a. Show that varying the action (1) with respect to the worldline leads to the Lorentz equation of motion

$$mu^\alpha\nabla_\alpha u^\beta = eF^{\beta\gamma}u_\gamma.$$

b. Show that varying the action (1) with respect to the 4-potential yields the Maxwell equation

$$\nabla_\alpha F^{\alpha\beta}(x^\gamma) = -4\pi j^\beta(x^\gamma),$$

where the 4-current density is

$$j^\beta(x^\gamma) = e\int d\tau\;\dot{x}^\beta(\tau)\frac{\delta^{(4)}[x^\gamma - x^\gamma(\tau)]}{\sqrt{-g[x^\gamma(\tau)]}}.$$

c. Show that when one computes the stress-energy that enters into Einstein's equations using the prescription

$$T^{\alpha\beta}(x) = \frac{2}{\sqrt{-g(x)}}\frac{\delta S_{\text{matter}}}{\delta g_{\alpha\beta}(x)},$$

one obtains $T^{\alpha\beta} = T^{\alpha\beta}_{\rm particle} + T^{\alpha\beta}_{\rm EM}$, where

$$T^{\alpha\beta}_{\rm particle}(x^\gamma) = m \int d\tau \; \frac{\dot{x}^\alpha(\tau)\dot{x}^\beta(\tau)}{\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}} \;\; \frac{\delta^{(4)}[x^\gamma - x^\gamma(\tau)]}{\sqrt{-g[x^\gamma(\tau)]}}$$

and

$$T^{\alpha\beta}_{\rm EM} = \frac{1}{4\pi}\left[F^{\alpha\gamma}F^{\beta}_{\ \gamma} - \frac{1}{4} g^{\alpha\beta} F_{\gamma\delta}F^{\gamma\delta}\right].$$

Note that here τ is any parameter along the worldline, not necessarily the particle's proper time.

3. *Action principle for a non-minimally-coupled scalar field:* [10 points] Consider the action for a scalar field $\Phi(x^\gamma)$

$$S = S_{\rm gravity} + S_{\rm matter},$$

where $S_{\rm gravity}$ is as given in problem 3 and

$$S_{\rm matter} = -\frac{1}{2} \int d^4x \sqrt{-g(x)} \;\; \left[g^{\alpha\beta} \nabla_\alpha \Phi \, \nabla_\beta \Phi + m^2 \Phi^2 + \xi R \Phi^2 \right]. \tag{1}$$

Here m is the mass of the scalar field, and the dimensionless constant ξ is called the curvature coupling constant. [The scalar field is *minimally coupled* if $\xi = 0$.]

a. Show that varying the action (1) with respect to the scalar field yields the equation of motion

$$\left[\nabla^\alpha \nabla_\alpha - m^2 - \xi R\right] \Phi = 0. \tag{2}$$

b. Compute from the action (1) the stress-energy tensor $T^{\alpha\beta}$ that enters into Einstein's equation, using the prescription of problem 3. Show that the result is

$$\begin{aligned} T_{\alpha\beta} &= \nabla_\alpha \Phi \nabla_\beta \Phi - \frac{1}{2} g_{\alpha\beta} (\nabla^\gamma \Phi \nabla_\gamma \Phi) - \frac{1}{2} g_{\alpha\beta} m^2 \Phi^2 \\ &+ \xi \left[G_{\alpha\beta} \Phi^2 - 2 \nabla_\alpha (\Phi \nabla_\beta \Phi) + 2 g_{\alpha\beta} \nabla^\gamma (\Phi \nabla_\gamma \Phi) \right]. \end{aligned} \tag{3}$$

Show that this stress-energy tensor is conserved when the equation of motion (2) is satisfied.

c. Specialize to Minkowski spacetime $g_{\alpha\beta} = \eta_{\alpha\beta}$, and compute the *canonical energy-momentum tensor*

$$T^{\alpha\beta}_{\rm can} \equiv -\partial^\alpha \Phi \, \frac{\partial \mathcal{L}}{\partial(\partial_\beta \Phi)} + \eta^{\alpha\beta} \mathcal{L},$$

where $\mathcal{L}$ is the Lagrangian density in (1). [This definition of energy momentum tensor is obtained by applying Noether's theorem to the translational symmetry of Minkowski spacetime.] Show that the result is

$$T^{\alpha\beta}_{\rm can} = \partial^\alpha \Phi \, \partial^\beta \Phi - \frac{1}{2} \eta^{\alpha\beta} (\partial^\gamma \Phi \, \partial_\gamma \Phi) - \frac{1}{2} \eta^{\alpha\beta} m^2 \Phi^2. \tag{4}$$

Note that this canonical stress energy tensor differs from (3) even when $g_{\alpha\beta} = \eta_{\alpha\beta}$. The correct energy momentum tensor to use in Einstein's equations in this case is the tensor (3).

d. Again specialize to Minkowski spacetime $g_{\alpha\beta} = \eta_{\alpha\beta}$, assume that the scalar field satisfies its equation of motion (2), and assume that $m = 0$. Show that the total mass-energy $\mathcal{E} \equiv \int d^3x T^{tt}$ given by integrating the energy density over all space is

$$\mathcal{E} = \frac{1}{2} \int d^3x \, \left[\dot{\Phi}^2 + (\nabla \Phi)^2 \right],$$

for *both* the GR energy momentum tensor (3) and the canonical energy momentum tensor (4).