Physics 6553 : Problem Set 12

Due Thursday, Nov 29, 2012

1. *Wormholes:* [5 points] Consider the maximally extended Schwarzschild solution discussed in lectures. Show that the equatorial 2-surface $t = \text{const}$, $\theta = \pi/2$ has a intrinsic geometry given in isotropic coordinates by

$$ds^2 = \left[1 + \frac{M}{2\bar{r}}\right]^4 \left[d\bar{r}^2 + \bar{r}^2 d\varphi^2\right].$$

Construct an embedding diagram for this geometry. Show that the region $0 < \bar{r} \ll M/2$ is an asymptotically flat space, and that the region $\bar{r} \gg M/2$ is another asymptotically flat space, and that the two spaces are connected by a tunnel ("wormhole") in the embedding diagram. For a discussion of why such wormholes almost certainly do not occur in the real Universe, see M. Morris, K. Thorne and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).

2. *Motion of charged particle in the Reissner-Nordstrom geometry and Cosmic Censorship:* [10 points]

a. Let M be a manifold and g_{ab} be a metric for which ξ^a is a killing vector field. Suppose that A_a is a vector potential such that

$$\left(\mathcal{L}_\xi A\right)_a \equiv \xi^b \nabla_b A_a + A_b \nabla_a \xi^b = 0.$$

[This expression is called the Lie derivative of A_a with respect to ξ^a.] Show that for a test particle of mass m, 4-velocity u^a and charge q whose motion is governed by the Lorentz force law, the quantity

$$\xi^a (m u_a + q A_a)$$

is a constant along the particles worldline.

b. The Reissner-Nordstrom spacetime is given by the metric and vector potential

$$ds^2 = -w(r)dt^2 + \frac{1}{w(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

and $A_a dx^a = -(Q/r)dt$, where $w(r) = 1 - 2M/r + Q^2/r^2$. Show that the Lie derivative of the vector potential with respect to the killing vector fields $\partial/\partial t$ and $\partial/\partial\varphi$ vanish. Deduce two conserved quantities for charged test-particle motion in the equatorial plane, and write down a one-dimensional effective potential governing general motion in the equatorial plane.

c. Consider a Reissner-Nordstrom black hole of mass M and charge Q with $Q < M$, together with a test particle of mass m, energy E and charge q. If the black hole can accrete the particle in the region of parameter space

$$Q + q > M + E \tag{1}$$

then the final black hole will have a charge greater than its mass, violating cosmic censorship. Show that this cannot occur by arguing as follows. Without loss of generality choose units such that $M = 1$. Let the charge to mass ratio of the test particle be $\hat{q}$, so that $q = \hat{q}m$. Let the dimensionless energy per unit mass of the particle be $\hat{E} = E/m$, and let the charge of the black hole be $Q = 1 - \hat{Q}m$, where $\hat{Q}$ is dimensionless. Write the effective potential governing the particles motion (when using proper time) in terms of the dimensionless parameters $\hat{E}$, $\hat{Q}$ and $\hat{q}$, the dimensionless angular momentum per unit mass, and the dimensionful parameter m. Argue that since the equation of motion is

$$u^b \nabla_b u^a = \hat{q} F^{ab} u_b + O(m),$$

that one should drop all terms of order $O(m)$ and $O(m^2)$ in the resulting expression for the effective potential. Drop these terms and analyze motion in the resulting effective potential to show that the black hole can never accrete the particle while satisfying the inequality $\hat{q} > \hat{E} + \hat{Q}$, which is the inequality (1) to first order in m.

3. *Instability of the inner horizon in Reissner-Nördstrom black holes.* [15 points]

a. *Null fluids:* Let $\vec{l}$ be a geodesic null vector field in any spacetime, and consider the stress energy tensor

$$T^{ab} = \rho l^a l^b. \tag{1}$$

Show that this stress energy tensor is conserved whenever

$$l^a \nabla_a \rho + \rho \nabla^a l_a = 0. \tag{2}$$

Show that the equation (2) reduces to an ordinary differential equation along each integral curve of the vector field $\vec{l}$, whose solution is

$$\rho(\lambda) \exp\left[\int d\lambda\, \theta(\lambda)\right] = \text{const},$$

where $\vec{l} = d/d\lambda$ and $\theta \equiv \nabla^a l_a$. The matter model (1) is called a null fluid. It can be a good approximation to the stress energy tensor of scalar fields or electromagnetic fields when the wavelengths of the radiation is small compared to the radius of spacetime curvature.

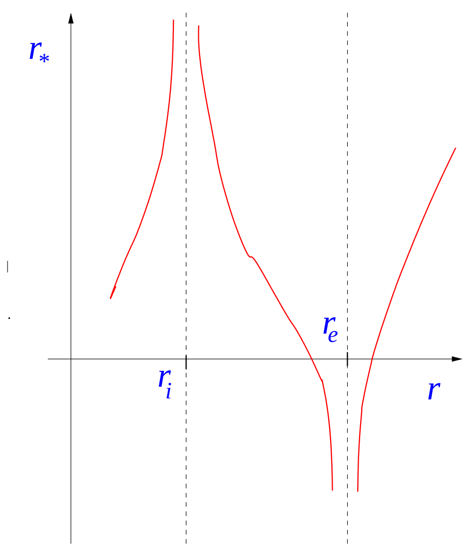

FIG 1

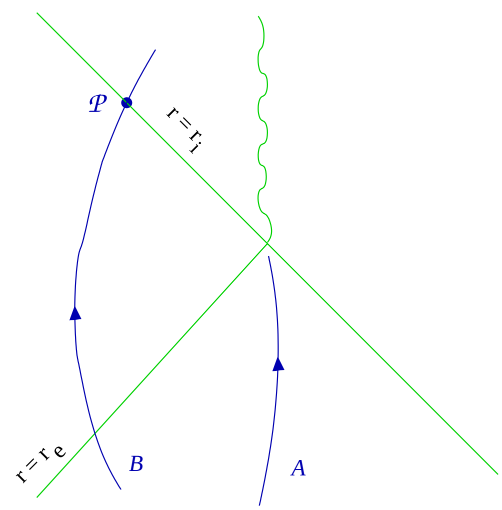

FIG 2

b. Starting from the usual form of the Reissner-Nórdstrom metric, show that it can be written as

$$ds^2 = -f(r) du dv + r^2 d\Omega^2,$$

where

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

$v = t + r^*$, $u = t - r^*$, and $r^* = r^*(r)$ is given by (see Fig. 1)

$$r^* = \int \frac{dr}{f(r)}.$$

The conformal diagram for this metric is sketched above (Fig 2). The event horizon is at $r = r_e$ and the inner horizon is at $r = r_i$, where $r_i < r_e$ and r_i, r_e are the two roots of $f(r) = 0$.

c. Show that for any two-dimensional spacetime of the form $ds^2 = \exp[\sigma(u,v)]dudv$, the vector fields $e^{-\sigma}\partial_u$ and $e^{-\sigma}\partial_v$ are geodesic. Deduce that in the Reissner-Nördstrom metric, the vector field

$$\vec{l} = \frac{1}{f}\frac{\partial}{\partial u}$$

is geodesic. Note that $l_\alpha dx^\alpha$ is simply $-dv/2$. Using part a., or otherwise, show that the stress-energy tensor

$$T_{ab} = \frac{L(v)}{4\pi r^2}\, l_a l_b \tag{3}$$

is conserved for all choice of functions $L(v)$. The model (3) is a good approximate description of radiation falling into a black hole, when the wavelength of the radiation is small. The function $L(v)$ is called a luminosity function; it is the power per unit "time" v being radiated into the black hole. When black holes are first formed, it's non-sphericities give rise to residual radiation (electromagnetic and gravitational) that falls into the black hole with $L(v)$ decaying like a power law,

$$L(v) \propto v^{-p}, \tag{4}$$

where p is some positive integer ≥ 3 [Price's theorem, exercise 32.10 of MTW].

d. Consider a static observer A at at some point $r = r_A$ outside the black hole (see Fig. 2). Show that for the model (3) of the radiation influx, the observer A measures an energy density given by

$$\rho_A(v) = \frac{L(v)}{4\pi r_A^2}\frac{1}{f(r_A)}$$

where $v = t + r^*(r_A)$. Clearly observer A measures a finite flux of energy at all times. Show that the total amount of energy that flows into the black hole is finite for the model (4) of the energy flux.

e. *Kruskal coordinates for the inner horizon:* At the inner or Cauchy horizon we have $r^* \to +\infty$. Show that $f(r)$ can be written as

$$f(r) = e^{-2\kappa_i r^*} g(r) \tag{5}$$

where $g(r)$ is smooth and non-vanishing in a neighborhood of $r = r_i$, and

$$\kappa_i = \frac{1}{2}|f'(r_i)|$$

is called the surface gravity of the inner horizon. Using Eq. (5) show that the Reissner-Nördstrom metric for $r_i < r < r_e$ can be written as $ds^2 = -g(r)dUdV/\kappa_i^2 + r^2 d\Omega^2$, where

$$V = -e^{-\kappa_i v}, \qquad U = e^{\kappa_i u}. \tag{6}$$

The Kruskal coordinates U and V are good spacetime coordinates in a neighborhood of $r = r_i$, unlike u and v.

f. Consider now a second, intrepid observer B who falls inside the black hole and crosses the inner horizon (see Fig. 2). Show that the energy density measured by B is

$$\rho_B = \frac{L(v)}{4\pi r^2}\left(\frac{dv}{d\tau}\right)^2,$$

where $dv/d\tau = u^v$ is one of the components of B's 4 velocity. Consider now the point $\mathcal{P}$ where B's world line intersects the inner horizon. At this point, $v \to \infty$, but V is finite. Argue that $dV/d\tau$ should be finite at $\mathcal{P}$, since the coordinate V is regular in a neighborhood of $\mathcal{P}$. Using Eq. (6), deduce that

$$\frac{dv}{d\tau} = ke^{\kappa_i v}\left[1 + O(e^{-\kappa_i v})\right].$$

for some constant k. Hence show that for large v,

$$\rho_B(v) \propto L(v)e^{2\kappa_i v}\left[1 + O(e^{-\kappa_i v})\right].$$

From Eq. (4), this energy density diverges as B approaches the point $\mathcal{P}$, even though $L(v) \to 0$ as $v \to \infty$. This instability is called the *blueshift instability* since B perceives ingoing quanta to be blueshifted to ever higher frequencies and energies as he approaches the inner horizon.

g. *Optional:* For those of you who are adventurous, try inserting the stress-tensor (3) as a source term into Einstein's equations and solving for the modification to the spacetime geometry. One finds that the Cauchy horizon becomes a null singularity. For details, see W. A. Hiscock, Phys. Lett. A **83**, 110 (1981).