Physics 6553 : Problem Set 4

Due Thursday, Sept 27, 2012

1. *The two sphere:* [5 points] Consider the unit two-sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ with coordinates (θ, φ).

a. Suppose that ∇ is a connection on the two-sphere for which the geodesics are all great circles [i.e., circles in a plane which intersects the origin $(x, y, z) = (0, 0, 0)$]. Determine the connection coefficients of ∇ in the coordinate system (θ, φ). Is this connection a flat connection?

b. Give an example of a flat connection in a local region on the two sphere.

c. **Optional:** Show that there is no globally defined, smooth, flat connection on the two sphere.

2. *Action principles for geodesics:* [10 points] Let $g_{\alpha\beta}$ be a metric on a manifold M, let $\mathcal{P}$ and $\mathcal{Q}$ be fixed points in M, and let $\mathcal{C}$ be a curve in M joining $\mathcal{P}$ to $\mathcal{Q}$ given in terms of coordinates x^α and a parameter s by $x^\alpha = x^\alpha(s)$. The length of the curve $\mathcal{C}$ is defined to be

$$l[x^\alpha(s)] = \int ds \sqrt{\epsilon\, g_{\alpha\beta}[\vec{x}(s)]\dot{x}^\alpha(s)\dot{x}^\beta(s)}, \tag{1}$$

where $\epsilon = +1$ or -1 is a constant depending on the character of the curve. In this problem we will show that curves which are local extrema of the length functional (1) are geodesics. Note that we cannot use the Euler-Lagrange equations in a straightforward way with the functional (1) due to reparameterization invariance, so we shall instead compute the variation of length directly.

a. Show that under a variation $x^\alpha(s) \to x^\alpha(s) + \delta x^\alpha(s)$, the first order change in length is given by

$$\delta l = \epsilon \int ds \left[\frac{g_{\alpha\beta,\gamma}\dot{x}^\alpha \dot{x}^\beta \delta x^\gamma + 2 g_{\alpha\beta}\dot{x}^\alpha \delta\dot{x}^\beta}{2\sqrt{\epsilon\, g_{\gamma\delta}\dot{x}^\gamma \dot{x}^\delta}} \right].$$

By performing an integration by parts, and assuming that the path variation keeps fixed the endpoints $\mathcal{P}$ and $\mathcal{Q}$, show that the variation in length can be written as $\delta l = \epsilon \int ds\, L_\gamma(s)\delta x^\gamma(s)$, where

$$L_\gamma = \frac{g_{\alpha\beta,\gamma}\dot{x}^\alpha \dot{x}^\beta}{2\sqrt{\epsilon\, g_{\delta\lambda}\dot{x}^\delta \dot{x}^\lambda}} - \frac{d}{ds}\left[\frac{g_{\alpha\gamma}\dot{x}^\alpha}{\sqrt{\epsilon\, g_{\delta\lambda}\dot{x}^\delta \dot{x}^\lambda}} \right].$$

b. Next, note that the formula obtained for δl is invariant under reparameterizations $s \to \bar{s}(s)$. Therefore, we can without loss of generality choose a parameterization for which $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$ is a constant along the curve. Show that for such a parameterization, the equation $L_\gamma = 0$ can be written as

$$\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma = 0,$$

the geodesic equation.

c. On a pseudo-Riemannian manifold, show that all three types of geodesic (timelike, null and spacelike) are local extrema of the quantity

$$\int ds\, g_{\alpha\beta}(x^\lambda(s)) \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}.$$

3. *Formulae for divergence and Laplacian:* [5 points]

a. Show that for any matrix $\mathbf{A}$ that $\det[\mathbf{1} + \varepsilon \mathbf{A}] = 1 + \varepsilon \mathrm{tr}\, \mathbf{A} + O(\varepsilon^2)$, where $\mathrm{tr}\, \mathbf{A}$ is the trace of $\mathbf{A}$ and $\det \mathbf{A}$ is the determinant of $\mathbf{A}$. Deduce that if $\mathbf{B}(\lambda)$ is a matrix that depends on a parameter λ, then

$$\frac{d}{d\lambda} \ln \det \mathbf{B}(\lambda) = \mathrm{tr}\left[\mathbf{B}^{-1}(\lambda) \cdot \mathbf{B}'(\lambda)\right].$$

b. Deduce from the above identity that the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ associated with a metric $g_{\alpha\beta}$ satisfy

$$\Gamma^\alpha_{\alpha\beta} = \frac{1}{2g}\frac{\partial g}{\partial x^\beta},$$

where $g = |\det g_{\alpha\beta}|$.

c. Deduce the formulae

$$\nabla_\alpha v^\alpha = \frac{1}{\sqrt{g}} \partial_\alpha (\sqrt{g} v^\alpha)$$

and

$$\nabla^\alpha \nabla_\alpha \Phi = \frac{1}{\sqrt{g}} \partial_\alpha \left[\sqrt{g} g^{\alpha\beta} \partial_\beta \Phi\right]$$

for any vector field v^α and any scalar field Φ.

4. *Rindler spacetime:* [10 points] Find the general solution of the geodesic equation in two-dimensional Rindler spacetime, where the coordinates are (t, x) and the metric is

$$ds^2 = -x^2 dt^2 + dx^2.$$