Physics 6553 : Problem Set 5

Due Thursday , Oct 4, 2012

1. *Curvature on a Non-coordinate Bases:* [10 points] Consider a manifold together with a set of vectors $\vec{e}_{\hat{\alpha}}$ that form a basis at every point. The components of the metric on this basis are $g_{\hat{\alpha}\hat{\beta}}$. The connection coefficients $\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}$ are defined by the equation $\nabla_{\vec{e}_{\hat{\alpha}}}\vec{e}_{\hat{\beta}} = \Gamma^{\hat{\gamma}}_{\hat{\beta}\hat{\alpha}}\vec{e}_{\hat{\gamma}}$, and the commutation coefficients $c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}$ are defined by $\left[\vec{e}_{\hat{\alpha}}, \vec{e}_{\hat{\beta}}\right] = c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}\vec{e}_{\hat{\gamma}}$.

a. Using the identity $\nabla_{\vec{u}}\vec{v} - \nabla_{\vec{v}}\vec{u} = [\vec{u}, \vec{v}]$, show that $c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} = \Gamma^{\hat{\gamma}}_{\hat{\beta}\hat{\alpha}} - \Gamma^{\hat{\gamma}}_{\hat{\alpha}\hat{\beta}}$.

b. By acting on $g_{\hat{\alpha}\hat{\beta}} = \vec{e}_{\hat{\alpha}} \cdot \vec{e}_{\hat{\beta}}$ with the differential operator $\nabla_{\vec{e}_{\hat{\gamma}}}$, show that

$$\Gamma^{\hat{\sigma}}_{\hat{\alpha}\hat{\gamma}}g_{\hat{\sigma}\hat{\beta}} + \Gamma^{\hat{\sigma}}_{\hat{\beta}\hat{\gamma}}g_{\hat{\sigma}\hat{\alpha}} = g_{\hat{\alpha}\hat{\beta},\hat{\gamma}},$$

where the quantity on the right hand side is $\vec{e}_{\hat{\gamma}}(g_{\hat{\alpha}\hat{\beta}})$.

c. By combining the results of parts a. and b., show that the connection coefficients are given by

$$\Gamma^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} = \frac{1}{2}g^{\hat{\alpha}\hat{\mu}}\left[g_{\hat{\mu}\hat{\beta},\hat{\gamma}} + g_{\hat{\mu}\hat{\gamma},\hat{\beta}} - g_{\hat{\beta}\hat{\gamma},\hat{\mu}} + c_{\hat{\mu}\hat{\beta}\hat{\gamma}} + c_{\hat{\mu}\hat{\gamma}\hat{\beta}} - c_{\hat{\beta}\hat{\gamma}\hat{\mu}}\right]$$

where $c_{\hat{\alpha}\hat{\beta}\hat{\gamma}} \equiv g_{\hat{\mu}\hat{\gamma}}c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\mu}}$. The first three terms in square brackets vanish for the case of an orthonormal basis, where $g_{\hat{\alpha}\hat{\beta}}$ are constants, and the last three terms vanish for the case of a coordinate basis, for which $c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}} = 0$.

d. Show that the components of the Riemann tensor are

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}}{}^{\hat{\delta}} = \Gamma^{\hat{\delta}}_{\hat{\gamma}\hat{\alpha},\hat{\beta}} - \Gamma^{\hat{\delta}}_{\hat{\gamma}\hat{\beta},\hat{\alpha}} + \Gamma^{\hat{\sigma}}_{\hat{\gamma}\hat{\alpha}}\Gamma^{\hat{\delta}}_{\hat{\sigma}\hat{\beta}} - \Gamma^{\hat{\sigma}}_{\hat{\gamma}\hat{\beta}}\Gamma^{\hat{\delta}}_{\hat{\sigma}\hat{\alpha}} + c_{\hat{\alpha}\hat{\beta}}{}^{\hat{\sigma}}\Gamma^{\hat{\delta}}_{\hat{\gamma}\hat{\sigma}}.$$

Note that, unlike the case for a coordinate basis, the ordering of the indices on the connection coefficients in this formula is important, since the connection coefficients are no longer symmetric.

[*Hint:* You should be able to do this problem without ever referring to a coordinate system or to coordinate components of tensors.]

2. [10 points] *Spherically symmetric spacetime:* Consider the spacetime with coordinates (t, r, θ, φ) and metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

We define an orthonormal basis by $\vec{e}_{\hat{t}} = e^{-\Phi}\partial_t$, $\vec{e}_{\hat{r}} = e^{-\Lambda}\partial_r$, $\vec{e}_{\hat{\theta}} = (1/r)\partial_\theta$ and $\vec{e}_{\hat{\varphi}} = (1/r\sin\theta)\partial_\varphi$. Show that the nonvanishing, orthonormal-basis components of the Einstein tensor are

$$G_{\hat{t}\hat{t}} = \frac{1}{r^2}\frac{d}{dr}\left[r(1 - e^{-2\Lambda})\right], \qquad G_{\hat{r}\hat{r}} = -\frac{1}{r^2}(1 - e^{-2\Lambda}) + \frac{2}{r}e^{-2\Lambda}\Phi'(r),$$

and

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\varphi}\hat{\varphi}} = \frac{1}{r}e^{-2\Lambda}\left[\Phi' + r(\Phi')^2 - \Lambda'(1 + r\Phi') + r\Phi''\right].$$

3. [10 points] *Geometric interpretation of the Riemann Tensor:* Let S be a two dimensional surface in a manifold parameterized by two coordinates s and t, so that $x^\alpha = x^\alpha(t,s)$. Let $\mathcal{P}$ be the point $s=0$, $t=0$, and for any $\varepsilon > 0$ consider the closed curve Γ_ε in M consisting of the four segments $x^\alpha = x^\alpha(u)$ for $0 \le u \le \varepsilon$ with

$$(i) \quad x^\alpha(u) = x^\alpha(u,0), \qquad (ii) \quad x^\alpha(u) = x^\alpha(\varepsilon,u),$$
$$(iii) \;\; x^\alpha(u) = x^\alpha(\varepsilon-u,\varepsilon), \qquad (iv) \quad x^\alpha(u) = x^\alpha(0,\varepsilon-u).$$

That is, Γ_ε starts at $\mathcal{P}$, moves along the $s=0$ curve, then along the $t=\varepsilon$ curve, then back along the $s=\varepsilon$ curve, then completes the quadrilateral back to $\mathcal{P}$. Let Λ_ε be the holonomy of the curve Γ_ε, that is, the linear map from $T_\mathcal{P}(M)$ to $T_\mathcal{P}(M)$ obtained by parallel transporting vectors around Γ_ε. Show that for any vector v^α at $\mathcal{P}$,

$$(\Lambda_\varepsilon \vec{v})^\alpha = v^\alpha + \varepsilon^2 R_{\gamma\beta\delta}{}^{\alpha}\, t^\gamma s^\beta v^\delta + O(\varepsilon^3),$$

where s^α is the vector $\partial/\partial s$ at $\mathcal{P}$ and t^α is the vector $\partial/\partial t$ at $\mathcal{P}$.

4. [5 points] *Properties of the Riemann tensor:* Consider tensors over an n-dimensional vector space V with $n \ge 3$. Let $\mathcal{S}$ be the space of $(0,4)$ tensors R_{abcd} satisfying the symmetries $R_{(ab)cd} = 0$ and $R_{abcd} = R_{cdab}$. In lecture we argued that the dimension of $\mathcal{S}$ is $N(N+1)/2$ where $N = n(n-1)/2$. Let $\mathcal{R}$ be the subspace of $\mathcal{S}$ consisting of tensors that satisfy the additional symmetry $R_{[abc]d} = 0$; the dimension of $\mathcal{R}$ is then the number of independent components of the Riemann tensor.

a. Show that $\mathcal{S} = \mathcal{R} \oplus \mathcal{T}$, where $\mathcal{T}$ is the space of completely antisymmetric $(0,4)$ tensors on V. [Hint: show that every element of $\mathcal{S}$ can be expressed uniquely as the sum of an element of $\mathcal{R}$ and an element of $\mathcal{T}$.]

b. Argue that the dimension of $\mathcal{T}$ is $n(n-1)(n-2)(n-3)/4!$, and deduce that the number of independent components of the Riemann tensor is $n^2(n^2-1)/12$.

c. Show that the Weyl tensor C_{abcd} defined by

$$R_{abcd} = C_{abcd} + \frac{2}{n-2}\left[g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right] - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b}$$

has the same symmetry properties as the Riemann tensor R_{abcd} and is traceless on all pairs of indices.