Physics 6553 : Problem Set 6

Due Thursday, Oct 11, 2012

1. *The Newtonian limit of general relativity:* [10 points] The Newtonian limit is described by a metric of the form

$$ds^2 = -\left[1 + \frac{2\Phi(t,\mathbf{x})}{c^2} + O\left(\frac{1}{c^4}\right)\right] c^2 dt^2 + O\left(\frac{1}{c^2}\right) dx^i dt + \left[\delta_{ij} + O\left(\frac{1}{c^2}\right)\right] dx^i dx^j, \qquad (1)$$

where $\Phi(t,\mathbf{x})$ is the Newtonian potential.

a. Show that the connection coefficients for this metric are of the form $\Gamma^\alpha_{\beta\gamma} = {}^{(0)}\Gamma^\alpha_{\beta\gamma} + O(c^{-2})$, where the Newtonian connection coefficients ${}^{(0)}\Gamma^\alpha_{\beta\gamma}$ are given by ${}^{(0)}\Gamma^i_{tt} = \partial_i \Phi$, with all the other components being zero.

b. By transforming from proper time derivatives to coordinate time derivatives, show that the equation for timelike geodesics can be written as

$$\frac{d^2 x^i}{dt^2} = -\Gamma^i_{tt} - 2\Gamma^i_{tj} v^j - \Gamma^i_{jk} v^j v^k + \left[\Gamma^t_{tt} + 2\Gamma^t_{tj} v^j + \Gamma^t_{jk} v^j v^k\right] v^i$$

where $v^i = dx^i/dt$. Deduce that the geodesic equation in the Newtonian connection ${}^{(0)}\Gamma^\alpha_{\beta\gamma}$ coincides with the equation of motion of Newtonian gravity.

c. Show that the components of the Ricci tensor are $R_{tt} = \nabla^2 \Phi + O(c^{-2})$, $R_{ti} = O(c^{-2})$, and $R_{ij} = O(c^{-2})$. Writing Einstein's equations in the form

$$R_{ab} = \frac{8\pi G}{c^4}\left(T_{ab} - \frac{1}{2} g_{ab} g^{cd} T_{cd}\right), \qquad (2)$$

and taking the leading order piece of the tt component, deduce the Poisson equation $\nabla^2\Phi = 4\pi G \rho$. Next, from the spatial components of (2), compute the leading order piece of the spatial components of the Ricci tensor in terms of Φ, and use this to show that the components of the Einstein tensor are $G^{tt} = 2\nabla^2\Phi/c^4$, $G^{ti} = O(c^{-4})$, $G^{ij} = O(c^{-4})$.

d. For a Newtonian fluid with density ρ, pressure p and 3-velocity v^i, argue that the components of the stress energy tensor are $T^{tt} = \rho$, $T^{ti} = \rho v^i$ and $T^{ij} = \rho v^i v^j + p\delta^{ij}$. [Hint: The stress tensor in the rest frame of the fluid is diagonal with diagonal elements ρ, p, p, p. Boost into the lab frame and take the $c \to \infty$ limit of the Lorentz transformation.] Using stress-energy conservation $\nabla_\alpha T^{\alpha\beta} = 0$, the Newtonian metric (1), and working to leading order in c^{-2} deduce the equations of Newtonian hydrodynamics: the continuity equation $\dot\rho + \partial_i(\rho v^i) = 0$ and Euler's equation $\dot v^i + v^j \partial_j v^i = -(\partial_i p)/\rho - \partial_i \Phi$.

2. *Geodesic Deviation in Newtonian Gravity:* [5 points] Consider two freely falling observers in Newtonian gravity, whose worldlines are $\mathbf{x} = \mathbf{x}_1(t)$ and $\mathbf{x} = \mathbf{x}_2(t)$. Let $\boldsymbol{\Delta}\mathbf{x}(t) \equiv \mathbf{x}_2(t) - \mathbf{x}_1(t)$ be the separation vector from one observer to the other.

a. Derive from Newtonian gravity the differential equation

$$\frac{d^2 \Delta x^i}{dt^2} = \mathcal{E}^i{}_j(t) \Delta x^j(t) \qquad (2)$$

which is accurate to first order in $\boldsymbol{\Delta}\mathbf{x}$, where the quantity $\mathcal{E}_{ij}$, called the tidal force tensor, is given by

$$\mathcal{E}_{ij}(t) = -\frac{\partial^2 \Phi}{\partial x^i \partial x^j}[\mathbf{x}_1(t), t]. \qquad (3)$$

b. We now re-derive this result from general relativity. Starting from the metric (1) in question 1 describing Newtonian gravitational fields, show that $R_{titj} = \Phi_{,ij} + O(v^{-2})$. Insert these Riemann tensor components and the metric (1) into the orthonormal-basis version of the geodesic deviation equation and expand to the leading order in ε to obtain equation (2).

3. *Gauge freedom in the Newtonian Limit:* [15 points] There is a subgroup of the full group of coordinate transformations that preserves the form (1) of the Newtonian metric. Starting from the coordinates (t, x^i) and the metric (1), we transform to another coordinate system $(\bar{t}, \bar{x}^i)$, and we assume that the metric in these coordinates takes the form

$$ds^2 = -\frac{1}{\varepsilon^2}\left[1 + 2\varepsilon^2\bar{\Phi}(\bar{t}, \bar{x}^j) + O(\varepsilon^4)\right]d\bar{t}^2 + O(\varepsilon^2)d\bar{x}^i d\bar{t} + \left[\delta_{ij} + O(\varepsilon^2)\right]d\bar{x}^i d\bar{x}^j, \tag{4}$$

where $\bar{\Phi}$ is the transformed Newtonian potential and $\varepsilon = 1/c$. The most general coordinate transformation that achieves this (up to constant displacements in time and to time-independent spatial rotations) is

$$x^i(\bar{t}, \bar{x}^j) = \bar{x}^i + z^i(\bar{t}) + O(\varepsilon^2), \qquad t(\bar{t}, \bar{x}^j) = \bar{t} + \varepsilon^2\left[\beta(\bar{t}) + \dot{z}^i(\bar{t})\bar{x}^i\right] + O(\varepsilon^4), \tag{5}$$

where $\beta(\bar{t})$ and $z^i(\bar{t})$ are an arbitrary functions of time. This represents a transformation to an accelerated reference frame. The transformations with $\ddot{z}^i = 0$ are Galilean transformations, parameterized by the constant velocity $\dot{z}^i$.

a. Show that the new potential is given by

$$\bar{\Phi}(\bar{t}, \bar{x}^j) = \Phi[\bar{t}, \bar{x}^j + z^j(\bar{t})] + \dot{\beta}(\bar{t}) + \ddot{z}^i(\bar{t})\bar{x}_i - \frac{1}{2}\dot{z}^i(\bar{t})\dot{z}_i(\bar{t}).$$

b. To derive the coordinate transformation (5), argue as follows. Let the coordinate transformation to zeroth order in ε be $x^i = x^i(\bar{t}, \bar{x}^j) + O(\varepsilon^2)$, $\quad t = t(\bar{t}, \bar{x}^j) + O(\varepsilon^2)$. Substituting this into the metric expansion (1), show that the leading order expression for the spatial metric is

$$-\frac{1}{\varepsilon^2}\frac{\partial t}{\partial \bar{x}^i}\frac{\partial t}{\partial \bar{x}^j}d\bar{x}^i d\bar{x}^j + O(1).$$

This is in conflict with the expansion (4) unless $\partial t/\partial \bar{x}^i = 0$. Similarly, the leading order expression for the time-time piece of the line element is

$$-\frac{1}{\varepsilon^2}\left(\frac{\partial t}{\partial \bar{t}}\right)^2 d\bar{t}^2 + O(1),$$

which disagrees with the expansion (4) unless $\partial t/\partial \bar{t} = \pm 1$. Assuming that the coordinate transformation preserves the time orientation and neglecting constant displacements in time, deduce that $t = \bar{t} + O(\varepsilon^2)$. Write this relation as $t = \bar{t} + \varepsilon^2\alpha(\bar{t}, \bar{x}^j) + O(\varepsilon^4)$, where the function $\alpha(\bar{t}, \bar{x}^j)$ is as yet undetermined.

c. Show that the leading order expression for the spatial metric is now

$$\delta_{kl}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}d\bar{x}^i d\bar{x}^j + O(\varepsilon^2) = \delta_{ij}d\bar{x}^i d\bar{x}^j + O(\varepsilon^2),$$

Deduce that, for each fixed $\bar{t}$, the function $x^i = x^i(\bar{t}, \bar{x}^j)$ is an isometry of 3-dimensional Euclidean space (i.e. a map that preserves the flat metric δ_{ij}). It follows that this map is thus of the form

$$x^i = R^i{}_j(\bar{t})\bar{x}^j + z^i(\bar{t}) + O(\varepsilon^2)$$

for some time-dependent rotation matrix $R^i{}_j(\bar{t})$ and some time-dependent displacement $z^i(\bar{t})$.

d. Show that the leading order expression for the space-time piece of the line element is now

$$\left\{2\delta_{ik}R^k{}_l(\bar{t})\left[\dot{R}^i{}_j(\bar{t})\bar{x}^j + \dot{z}^i(\bar{t})\right] - 2\frac{\partial \alpha}{\partial \bar{x}^l}\right\}d\bar{t}d\bar{x}^l + O(\varepsilon^2).$$

and argue that the first term here must vanish in order to be compatible with (4), which gives

$$\delta_{ik}R^k{}_l(\bar{t})\left[\dot{R}^i{}_j(\bar{t})\bar{x}^j + \dot{z}^i(\bar{t})\right] = \frac{\partial \alpha}{\partial \bar{x}^l}. \tag{6}$$

Show that if $\dot{R}^i{}_j(\bar{t})$ is non-vanishing, it is impossible to find any function $\alpha(\bar{t}, \bar{x}^j)$ which satisfies this equation. Deduce that the rotation matrix is time-independent, and specialize the new coordinate system $\bar{x}^i$ so that $R^i{}_j = \delta^i_j$. Finally solve Eq. (6) for $\alpha(\bar{t}, \bar{x}^j)$ to obtain the coordinate transformation (5).