Physics 6553 : Problem Set 7

Due Thursday, Oct 18, 2010

1. *Birkhoff's Theorem:* [10 points] In lecture we showed that the Schwarzschild solution is the only static, spherically symmetric, vacuum solution of the Einstein equations. In this problem you will generalize this result to show that it is in fact the only spherically symmetric, vacuum solution (Birkhoff's theorem).

  a. Argue that one can find coordinates (t, r, θ, ϕ) such that the metric has the form

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + e^{2\Lambda(t,r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

  This is the same as the metric used in lecture except that the potentials Φ and Λ are now allowed to be time dependent.

  b. For the above metric, the nonvanishing components of the Einstein tensor are

$$G_{tt} = \frac{e^{2\Phi-2\Lambda}\left(2r\Lambda_r + e^{2\Lambda} - 1\right)}{r^2}, \quad G_{tr} = \frac{2\Lambda_t}{r}, \quad G_{rr} = \frac{2r\Phi_r - e^{2\Lambda} + 1}{r^2},$$

$$G_{\theta\theta} = e^{-2(\Lambda+\Phi)}r\left\{re^{2\Phi}\Phi_r{}^2 + e^{2\Phi}\Phi_r - e^{2\Phi}\Lambda_r\left(r\Phi_r + 1\right) + r\left[e^{2\Phi}\Phi_{rr} - e^{2\Lambda}\left(\Lambda_t{}^2 - \Phi_t\Lambda_t + \Lambda_{tt}\right)\right]\right\},$$

  and $G_{\phi\phi} = \sin^2\theta\, G_{\theta\theta}$, where subscripts mean derivatives with respect to the corresponding variables. Show that the only solution to $G_{\alpha\beta} = 0$ is the Schwarzschild solution, up to gauge.

2. *Gravitational Field of a Thin Plane:* [10 points] In this problem you will find the general static, plane parallel solution of the vacuum Einstein equations, appropriate to describe the gravitational field of an infinite, thin plane.

  a. Argue that one can choose coordinates (t, x, y, z) such that the metric takes the form

$$ds^2 = -e^{2\alpha(x)}dt^2 + dx^2 + e^{2\gamma(x)}(dy^2 + dz^2).$$

  b. Show that the non-vanishing components of the Einstein tensor are

$$G_{tt} = -e^{2\alpha}\left(3\gamma'^2 + 2\gamma''\right), \quad G_{xx} = \gamma'\left(2\alpha' + \gamma'\right), \quad G_{yy} = G_{zz} = e^{2\gamma}\left(\alpha'^2 + \gamma'\alpha' + \gamma'^2 + \alpha'' + \gamma''\right).$$

  c. Show that one class of solutions is $\alpha = \ln(x - x_0) + \alpha_0$, $\gamma = \gamma_0$, where x_0, α_0, and γ_0 are constants. Show that these solutions are just flat, Minkowski spacetime, and thus can be discarded.

  d. Show that the general solution not of the above type is $\alpha = -\ln(x-x_0)/3 + \alpha_0$, $\gamma = 2\ln(x-x_0)/3 + \gamma_0$, where x_0, γ_0 and α_0 are constants. This is the Taub solution [Taub, A. H. (1951). Ann. Math., **53**, 472.].

  e. Write down the corresponding Newtonian solution, and describe the degree to which the Newtonian solution is a limiting case of the relativistic one.

Q3. *Isotropic coordinates for spherical spacetimes:* [5 points] The Schwarzschild coordinate system (t, r, θ, φ) for a spherically symmetric, static spacetime, discussed in the lectures, is given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2. \tag{1}$$

An alternative set of coordinates is the so-called isotropic coordinate system $(t, \bar{r}, \theta, \varphi)$, in which the metric takes the form

$$ds^2 = -e^{2\Phi(\bar{r})}dt^2 + e^{2\mu(\bar{r})}\left[d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2\sin^2\theta\, d\varphi^2\right], \tag{2}$$

for some potentials Φ and μ. Here $\Phi(\bar{r})$ means $\Phi(r)$ evaluated at $r = r(\bar{r})$.

  a. Starting from the metric (1), find a coordinate transformation $\bar{r} = \bar{r}(r)$ such that the metric in the new coordinates takes the form (2).

  b. Apply this transformation to the Schwarzschild metric to find the isotropic form of the Schwarzschild solution.