Physics 6553 : Problem Set 8

Due Thursday, Oct 25, 2012

1. *Motion in the Schwarzschild Geometry:* [10 points]

a. An stationary observer at $r = R$ in Schwarzschild coordinates shoots a projectile radially outward. The initial velocity of the projectile as measured by the observer is v. How large does v have to be in order for the projectile to escape to infinity?

b. Show that the angular velocity Ω of circular orbits of radius r, as measured by stationary observers at infinity, is given by the same formula as in Newtonian gravity, $\Omega^2 = M/r^3$.

c. Show that the location of the innermost stable circular orbit is at $r = 6M$.

d. For a particle of mass μ in a circular orbit about a spherical source of mass M, compute the conserved energy E of the orbit as a function of the frequency Ω of part b.

2. *Coordinate Transformations:* [5 points] Prove that the metric

$$ds^2 = -dt^2 + \frac{4}{9}\left[\frac{9M}{2(r-t)}\right]^{2/3} dr^2 + \left[\frac{9M}{2}(r-t)^2\right]^{2/3} d\Omega^2$$

(which looks dynamical because the metric coefficients depend on t) is actually static. Show that it is in fact the Schwarzschild geometry.

3. *Falling into a black hole:* [10 points]

An observer A falls radially in the Schwarzschild metric, starting from rest at $r = \infty$. As he falls he broadcasts a description of what he experiences using radio waves. Another observer B is stationary at $r = R$ with $R \gg M$, and monitors A's broadcasts. In this problem you will show that as A approaches the surface $r = 2M$, B sees A's broadcasts to become enormously redshifted, with the observed frequency ω_B varying with B's proper time t_B as

$$\omega_B \propto \exp\left[-\frac{t_B}{4M}\right]. \tag{2}$$

a. Show that the equation for the path $r(\tau)$ of observer A is $(dr/d\tau)^2 = 2M/r$. Without loss of generality take $\tau = 0$ as the observer crosses $r = 2M$. We only need to solve for the motion of the observer near $r = 2M$, and for this purpose we can use a Taylor expansion. Substitute the ansatz

$$r(\tau) = 2M - \gamma\tau + O(\tau^2) \tag{3}$$

into the equation of motion and show that the parameter $\gamma = 1$. Note that τ is negative while A is outside $r = 2M$.

b. Show that the equation of motion for $t(\tau)$ for observer A is $dt/d\tau = 1/(1 - 2M/r)$. Using the solution (3) in this equation, show that

$$t(\tau) = -2M \ln\left(-\frac{\tau}{M}\right) + t_0 + O(\tau),$$

where t_0 is a constant.

c. The radio waves are carried by photons which travel along radial null geodesics with $\theta = \theta_0$ and $\varphi = \varphi_0$ fixed. Let the geodesic which starts at $(t(\tau), r(\tau), \theta_0, \varphi_0)$ intersect the worldline of B at $(t_B, R, \theta_0, \varphi_0)$. We would like to compute B's proper time t_B as a function of τ. Show that along any radial null geodesic, the quantity $t - r_*(r)$ is conserved, where

$$r_*(r) = \int \frac{dr}{1 - 2M/r} = r + 2M \ln\left|\frac{r}{2M} - 1\right|.$$

Deduce that $t_B - r_*(R) = t(\tau) - r_*(r(\tau))$, and that

$$t_B(\tau) = t_1 - 4M \ln\left(-\frac{\tau}{M}\right) + O(\tau),$$

where t_1 is a constant. Invert this relation to express τ as a function of t_B, and argue that the frequency measured by B is proportional to $d\tau/dt_B$. Hence deduce the formula (2).