Physics 6553 : Problem Set 9

Due Thursday, Nov 1, 2012

1. *Hydrodynamics of relativistic fluids:* [10 points] The stress-energy tensor for a relativistic fluid in terms of the 4-velocity u^a, pressure p and density ρ is

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab}, \tag{1}$$

and the local law of conservation of 4-momentum is

$$\nabla_a T^{ab} = 0. \tag{2}$$

a. By contracting (2) with the four velocity, and by also using the law of conservation of particle number $\nabla_a(nu^a) = 0$, where n is the number density of particles, deduce

$$\frac{d\rho}{d\tau} - \frac{\rho + p}{n}\frac{dn}{d\tau} = 0, \tag{3}$$

where $d/d\tau = u^a \nabla_a$ is derivative with respect to proper time along a fluid elements worldline.

b. By contracting the conservation law (2) with the projection tensor $h_{bc} = g_{bc} + u_b u_c$, deduce that

$$(\rho + p)a_c + \nabla_c p + u_c \frac{dp}{d\tau} = 0, \tag{4}$$

where a_c is the 4-acceleration.

c. Consider steady flows in the absence of gravity, for which there is some Lorentz frame (t, x^i) for which all the hydrodynamic variables are independent of t. Evaluate the time component of (4) in this frame, and combine with (3) to show that

$$\frac{d}{d\tau}\left[u^t \left(\frac{\rho + p}{n}\right)\right] = 0.$$

Show that the non-relativistic limit of this result is Bernouilli's theorem: that the quantity $\mathbf{v}^2/2 + u + p/\rho_M$ is conserved along flow lines, where u is the internal energy per unit mass and ρ_M is the mass density [related to ρ and n by $\rho = \rho_M(1 + u)$ and $\rho_M = mn$].

2. *Conservation of Energy for Newtonian Fluids:* [5 points] The energy density of a Newtonian fluid, neglecting gravity, is

$$\mathcal{E} = \rho_M \left(\frac{1}{2}\mathbf{v}^2 + u\right)$$

where u is the internal energy per unit mass, ρ_M is mass density and $\mathbf{v}$ is velocity. One might expect the energy flux to be be $\mathcal{F} = \mathcal{E}\mathbf{v}$. However this expression omits an important contribution. Consider an element of surface area dA orthogonal to the fluid velocity $\mathbf{v}$. A fluid element that crosses dA during a time dt moves through a distance $dl = vdt$, and as it moves, the fluid behind this element exerts a force pdA on it. That force, acting through the distance dl, feeds an energy $dE = (pdA)dl = pvdAdt$ across dA; the corresponding energy flux across dA has magnitude $dE/(dAdt) = pv$ and points in the $\mathbf{v}$ direction, so it contributes $p\mathbf{v}$ to the energy flux $\mathcal{F}$. Thus the total energy flux is

$$\mathcal{F} = \rho_M \mathbf{v}\left(\frac{1}{2}\mathbf{v}^2 + u + p/\rho_M\right).$$

In this problem you will derive the local law of energy conservation

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \mathcal{F} = \rho_M T \frac{ds}{dt}, \tag{2}$$

where $d/dt = \partial/\partial t + \mathbf{v} \cdot \nabla$ is the comoving time derivative, T is temperature and s is entropy per unit mass.

a. By combining the Euler and continuity equations, show that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_M v^2\right) + \nabla \cdot \left(\frac{1}{2}\rho_M v^2 \mathbf{v}\right) = -(\mathbf{v} \cdot \nabla)p.$$

b. By combining the first law of thermodynamics in the form $du = Tds - pd(1/\rho_M)$ with the definition $h = u + p/\rho_M$ of the enthalpy h per unit mass, show that $dp = \rho_M dh - \rho_M T ds$.

c. Use the result of part b to eliminate the gradient of pressure term on the right hand side of part a. Now use the continuity equation and the first law of thermodynamics again to derive the energy conservation equation (2).

3. *Numerical Models of Neutron Stars:* [10 points] Neutron stars are configurations of cold matter at the endpoint of thermonuclear evolution having central densities roughly in the range $10^{14}\,\mathrm{g\,cm^{-3}} < \rho < 10^{16}\,\mathrm{g\,cm^{-3}}$. In this density range (which is about the density of an atomic nucleus), the bulk of the material in the star can be roughly approximated as a degenerate Fermi gas of neutrons. The equation of state for such a gas takes the form

$$p = \frac{3^{2/3}\pi^{4/3}\hbar^2}{5m_{\mathrm{n}}^{8/3}}\rho^{5/3},$$

where $\hbar$ is the reduced Planck constant and m_{n} is the neutron mass. Integrate the Tolman-Oppenheimer-Volkoff equation numerically using Mathematica, Maple, or whatever else you prefer. Compute the total mass M (in Solar Masses) and the total radius R (in km) for a sequence of central densities ρ_{c}; e.g., $\rho_{\mathrm{c}} = 10^{14}, 3 \times 10^{14}, 10^{15}, 3 \times 10^{15}$, and $10^{16}\,\mathrm{g\,cm^{-3}}$. Plot the resulting mass-radius relation (analogous to Fig. 6.1 of Wald) for stellar models based on this equation of state.