Physics 7683 : Problem Set 1

Due Thursday, Sept 17, 2009

1. *Bogolubov Transformations for Special Cases:* Consider the time dependent harmonic oscillator $\hat{H} = \hat{p}^2/2 + \omega(t)^2\hat{q}^2/2$, where $\omega(t)$ is a smooth function taking the values $\omega(t) = \omega_{\rm in}$ at early times and $\omega(t) = \omega_{\rm out}$ at late times. The Bogolubov coefficients are defined by taking the solution $q(t) = \exp[-i\omega_{\rm in}t]/\sqrt{2\omega_{\rm in}}$ which is purely positive frequency at early times, and writing it at late times as a linear combination of positive and negative frequency solutions: $\sqrt{2\omega_{\rm out}}q(t) = \alpha^* \exp[-i\omega_{\rm out}t] - \beta \exp[i\omega_{\rm out}t]$. In this problem we will compute the coefficients in some special cases.

a. Consider the adiabatic regime $\dot{\omega}/\omega^2 \ll 1$. By performing a WKB type analysis show that the leading order solution that is purely positive frequency at early times is

$$q(t) = \frac{1}{\sqrt{2\omega(t)}} \exp\left[-i\int^t dt' w(t')\right].$$

Deduce that in this approximation the transformation is trivial, $\beta = 0$. Does this result continue to hold when one computes the subleading WKB (post adiabatic) corrections? [*Hint:* Replace ω by ω/ϵ in the differential equation, use an ansatz of the form $q(t) = [A(t) + \epsilon B(t) + \ldots] \exp[i\phi(t)/\epsilon]$, and expand the differential equation in powers of ϵ.]

b. Suppose that the frequency can be written as

$$\omega(t) = \omega_{\rm in} + \Delta\omega(t),$$

where the frequency perturbation is small, $\Delta\omega \ll \omega_{\rm in}$ and also $T\Delta\omega \ll 1$ where T is the duration of the period of time evolution. Derive an expression for $|\beta|$ in terms of the Fourier transform of $\Delta\omega$. Use your result to argue that if $\omega(t)$ is smooth, then $|\beta|$ goes to zero faster than any power of $1/(\omega_{\rm in}\tau)$ in the limit $\tau \to \infty$, where τ is the timescale over which $\omega(t)$ changes. Also argue that if any finite-order derivative of $\omega(t)$ has a discontinuity, then $|\beta|$ will scale as a power law $\propto 1/(\omega_{\rm in}\tau)^n$ for some finite integer n.

c. Suppose that the frequency $w(t)$ changes instantaneously from $\omega_{\rm in}$ to $\omega_{\rm out}$ at $t = 0$. Show that for this case the Bogolubov coefficients are given by

$$\alpha = \frac{1}{2}\left(\sqrt{\frac{\omega_{\rm in}}{\omega_{\rm out}}} + \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}}\right), \qquad \beta = \frac{1}{2}\left(\sqrt{\frac{\omega_{\rm in}}{\omega_{\rm out}}} - \sqrt{\frac{\omega_{\rm out}}{\omega_{\rm in}}}\right).$$

2. *Normal Ordered Form of Squeezing Operator:* Consider the Hilbert space of a harmonic oscillator whose annilihation operator is $\hat{a}$. In this problem you will derive the normal ordered form of the squeezing operator $\hat{S}$ which is defined by the property

$$\hat{S}^\dagger \hat{a} \hat{S} = \alpha\hat{a} + \beta^* \hat{a}^\dagger, \qquad (1)$$

where α and β are complex numbers. In other words, you will derive the function $\bar{f}^{(n)}(\mu, \mu^*)$ of a complex variable μ and its complex conjugate μ^* for which

$$\hat{S} = \; : \bar{f}^{(n)}(\hat{a}, \hat{a}^\dagger) : \; .$$

a. Using the fact that coherent states $|\mu)$ are eigenstates of the annihilation operator, argue that

$$\bar{f}^{(n)}(\mu, \mu^*) = (\mu| \hat{S} |\mu) \, .$$

b. Show that for any function g, we have $\left[g(\hat{a}), \hat{a}^\dagger\right] = g'(\hat{a})$.

c. Write hermitian conjugate of the defining relation (1) in the form $\hat{a}^\dagger \hat{S} = \hat{S}(\alpha^* \hat{a}^\dagger + \beta \hat{a})$, multiply on the left by $(\mu|$ and on the right by $|\mu)$. Show using parts a. and b. that this gives the differential equation

$$\mu^* \bar{f}^{(n)} = \beta \mu \bar{f}^{(n)} + \alpha^* \left(\mu^* + \frac{\partial}{\partial \mu} \right) \bar{f}^{(n)}.$$

Similarly from $\hat{a}\hat{S} = \hat{S}(\alpha \hat{a} + \beta^* \hat{a}^\dagger)$ derive the differential equation

$$\left(\mu + \frac{\partial}{\partial \mu^*} \right) \bar{f}^{(n)} = \alpha \mu \bar{f}^{(n)} + \beta^* \left(\mu^* + \frac{\partial}{\partial \mu} \right) \bar{f}^{(n)}.$$

d. Solve this pair of differential equations using an ansatz of the form

$$\bar{f}^{(n)} = \mathcal{N} \exp\left[A\mu^2 + B\mu\mu^* + C(\mu^*)^2 \right],$$

where $\mathcal{N}$, A, B and C are constants. Deduce the value of the normalization constant $\mathcal{N}$ from $1 = \langle 0 | \hat{S}^\dagger \hat{S} | 0 \rangle$ as in lecture. Thereby deduce that

$$\hat{S} = |\alpha|^{-1/2} : \exp\left[-\frac{\beta}{2\alpha^*}\hat{a}^2 + \frac{\beta^*}{2\alpha^*}\hat{a}^{\dagger\, 2} + \left(\frac{1}{\alpha^*} - 1 \right) \hat{a}^\dagger \hat{a} \right] : \; .$$

3. *Time-dependent, Driven Harmonic Oscillator:* In this problem we will generalize the analysis given in lecture to an oscillator which is driven in addition to having a time-varying frequency. The Hamiltonian of the system is

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega(t)^2 \hat{q}^2 - J(t)\hat{q}.$$

Assume that the source $J(t)$ vanishes and that the frequency $\omega(t)$ is constant at early times and at late times, with values $\omega_{\rm in}$ and $\omega_{\rm out}$. For any state $|\psi\rangle_{\rm in}$ defined on the (Heisenberg picture) in basis, define a corresponding state $|\psi\rangle_{\rm out}$ defined on the out basis by replacing in by out everywhere in the definition of the state. Then we have

$$|\psi\rangle_{\rm in} = \hat{S}^\dagger |\psi\rangle_{\rm out} \, ,$$

where $\hat{S}$ is a generalization of the operator derived in lecture. Derive an expression for $\hat{S}$ in terms of squeeze operators, rotation operators and displacement operators.