Physics 7683 : Problem Set 2

Due Thursday, Oct 1, 2009

1. *Bogolubov Transformations for Fermions:* Consider a Hilbert space of N fermionic oscillators with annilihation and creation operators $\hat{a}_i$ and $\hat{a}_i^\dagger$. These operator satisfy the anticommutation relations

$$\{\hat{a}_i, \hat{a}_j\} = 0, \qquad \left\{\hat{a}_i, \hat{a}_j^\dagger\right\} = \delta_{ij}, \qquad \left\{\hat{a}_i^\dagger, \hat{a}_j^\dagger\right\} = 0,$$

for $1 \leq i, j \leq N$. Define a new set of operators $\hat{b}_i$ by

$$\hat{b}_i = \alpha_{ij}\hat{a}_j + \beta_{ij}\hat{a}_j^\dagger$$

where α and β are complex, $N \times N$ matrices.

a. Show that the operators $\hat{b}_i$ satisfy the standard anticommutation relations if and only if the matrices α and β satisfy

$$1 = \alpha \cdot \alpha^\dagger + \beta \cdot \beta^\dagger, \qquad 0 = \beta \cdot \alpha^{\mathrm{T}} + \alpha \cdot \beta^{\mathrm{T}},$$

where $\dagger$ denotes the complex conjugate transpose of a matrix and T denotes transpose.

b. Specialize to the case $N = 1$ and show that the only solutions to these equations for the Bogolubov coefficients consist of multiplying by constant phases and/or interchanging $\hat{a}$ and $\hat{a}^\dagger$. Thus, there are no nontrivial solutions for $N = 1$, unlike the situation for bosons.

c. By counting parameters, argue that there is a 6 parameter set of solutions in the case $N = 2$. Show in particular that

$$\hat{b}_1 = \cos\psi\hat{a}_1 - \sin\psi\hat{a}_2^\dagger, \qquad \hat{b}_2 = -\cos\psi\hat{a}_2 - \sin\psi\hat{a}_1^\dagger$$

is a Bogolubov transformation. For this case, can you find an expression for the operator $\hat{S}$ that implements the Bogolubov transformation, i.e. satisfies

$$\hat{S}^\dagger\hat{a}_i\hat{S} = \hat{b}_i.$$

2. *Thermal Initial State:* In lecture we showed that if a (bosonic) harmonic oscillator with time-dependent frequency $\omega(t)$ is initially in its vacuum state, then the expected number of quanta present at late times is $N = |\beta|^2$, where β is the Bogolubov coefficient. Suppose that, instead, the initial state is a thermal state with temperature T. Compute N in terms of α, β, T, ω_{in} and ω_{out}. [*Hint:* compute the Wigner function representation of the state and how it transforms.]

3. *Canonical Commutation Relations:* Show using the mode expansion for a free scalar field in flat spacetime that the canonical, equal time commutation relations for the fields

$$\left[\hat{\Phi}(t,\mathbf{x}), \hat{\Phi}(t,\mathbf{y})\right] = 0, \qquad \left[\hat{\Phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})\right] = i\delta^3(\mathbf{x}-\mathbf{y}), \qquad [\hat{\pi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = 0$$

are satisfied if and only if the creation and annihilation operators for the plane wave modes satisfy the standard commutation relations

$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0, \qquad \left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger\right] = \delta^3(\mathbf{k}-\mathbf{k}'), \qquad \left[\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger\right] = 0.$$