

Physics 7683 : Problem Set 3

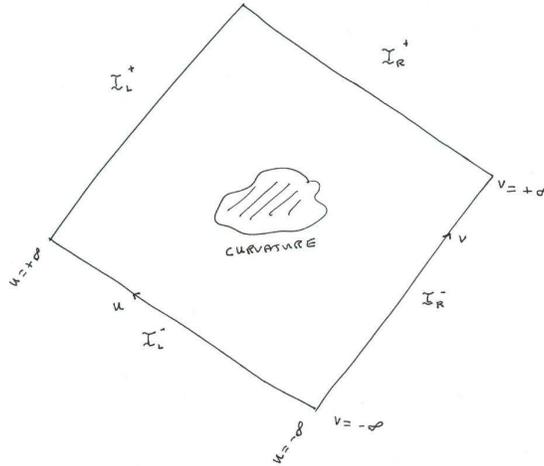
Due Thursday, Oct 15, 2009

1. Amplitude to Create $2n$ particles: Consider the situation discussed in lecture, where a spacetime is asymptotically stationary in the past, with Hilbert space \mathcal{H}_{in} , and also asymptotically stationary in the future, with Hilbert space \mathcal{H}_{out} . Starting with the in-vacuum state, show that the amplitude to create $2n$ particles is given by

$${}_{\text{out}} \langle i_1 i_2 \dots i_{2n}, \text{out} | 0, \text{in} \rangle_{\text{out}} = {}_{\text{out}} \langle 0, \text{out} | 0, \text{in} \rangle_{\text{out}} [V_{i_1 i_2} V_{i_3 i_4} \dots V_{i_{2n-1} i_{2n}} + \dots], \quad (1)$$

where $V = -\beta^* \alpha^{-1}$, $|i_1 i_2 \dots i_{2n}, \text{out} \rangle_{\text{out}} = \hat{a}_{i_1}^\dagger \hat{a}_{i_2}^\dagger \dots \hat{a}_{i_{2n}}^\dagger |0, \text{out} \rangle_{\text{out}}$, and otherwise the notation is the same as in lectures. The expression in the square brackets on the right hand side of (1) means that we divide the $2n$ integers i_1, i_2, \dots, i_{2n} into n pairs. There are $(2n)!/(n!2^n)$ ways of doing so, and we sum over all choices of pairs. Each pair is assigned to a single factor of V . Thus there are $(2n)!/(n!2^n)$ terms in the sum in the square brackets.

2. Particle Creation in Two Dimensions: In four dimensions, it is difficult to compute Bogolubov transformations explicitly except in spacetimes with symmetries. However two dimensions are sufficiently simple that it is possible to obtain an analytic expression for particle creation in a general dynamical geometry.



We write the 2D metric in the conformally flat form

$$ds^2 = -e^{2\sigma(u,v)} du dv, \quad (2)$$

where $-\infty < u < \infty$, $-\infty < v < \infty$. We specialize, for simplicity, to a spacetime where the curvature is confined to the compact region $u_1 < u < u_2$ and $v_1 < v < v_2$. There are four asymptotic regions for this spacetime, as shown in the Penrose diagram: (i) left past null infinity, \mathcal{I}_L^- , at $v = -\infty$; (ii) right past null infinity, \mathcal{I}_R^- , at $u = -\infty$; (iii) left future null infinity, \mathcal{I}_L^+ , at $u = \infty$; and (iv) right future null infinity, \mathcal{I}_R^+ , at $v = \infty$.

- a. Since the spacetime is flat in the past, in the vicinity of \mathcal{I}_L^- and \mathcal{I}_R^- , we can choose the coordinates so that $\sigma = 0$ there. However, σ will not be zero where there is curvature, and also to the future of where there is curvature. Show that σ is given by

$$\sigma(u, v) = -\frac{1}{2} \int_{-\infty}^u du' \int_{-\infty}^v dv' R_{uv}(u', v'), \quad (3)$$

where R_{uv} is the uv component of the Ricci tensor. Deduce that in the vicinity of right future null infinity \mathcal{I}_R^+ , the conformal factor is independent of v , $\sigma(u, v) = \sigma_\infty(u)$, where $\sigma_\infty(u)$ is the expression (3) evaluated at $v = \infty$.

- b. Consider now a free, massless, real scalar field Φ on this spacetime. Show that the general solution to the classical equation of motion for Φ is

$$\Phi(u, v) = \Phi_R(u) + \Phi_L(v), \quad (4)$$

for some real functions Φ_R and Φ_L , which describe right-moving radiation and left-moving radiation, respectively. Thus, the right moving and left moving sectors of the theory are decoupled from one another, even in the presence of curvature. For simplicity, we will consider only the right moving sector.

- c. By integrating the Klein-Gordon conserved current over the null surface $v = v_0$ for some constant v_0 , show that the Klein-Gordon inner product of two complex solutions $f(u)$, $g(u)$ in the right moving sector is

$$\langle f, g \rangle = i \int_{-\infty}^{\infty} du \left[f(u)^* g_{,u}(u) - f_{,u}(u)^* g(u) \right]. \quad (5)$$

- d. We assume that the field starts out in the “in vacuum” state in the vicinity of \mathcal{I}_L^- . Show that a suitable mode expansion of the field operator near \mathcal{I}_L^- is

$$\hat{\Phi}_R(u) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega u} \hat{a}_\omega + e^{i\omega u} \hat{a}_\omega^\dagger \right], \quad (6)$$

where $[\hat{a}_\omega, \hat{a}_{\omega'}^\dagger] = \delta(\omega - \omega')$. The in vacuum state $|0, \text{in}\rangle$ is defined by $\hat{a}_\omega |0, \text{in}\rangle = 0$ for all ω .

- e. Consider next measurements made near right future null infinity \mathcal{I}_R^+ . Show that the metric in this region can be written as $ds^2 = -dU dv$, where

$$U = \int_{-\infty}^u du' e^{2\sigma_\infty(u')}. \quad (7)$$

We can write down a mode expansion of the field operator with respect to the U coordinate, which are the modes relevant for measurements made near \mathcal{I}_R^+ . This mode expansion is

$$\hat{\Phi}_R(U) = \frac{1}{\sqrt{2\pi}} \int_0^\infty d\omega \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega U} \hat{b}_\omega + e^{i\omega U} \hat{b}_\omega^\dagger \right]. \quad (8)$$

The out vacuum state $|0, \text{out}\rangle$ is defined by $\hat{b}_\omega |0, \text{out}\rangle = 0$ for all ω . [Note that here we are implicitly identifying the in and out Hilbert spaces in the natural way, unlike the treatment in lectures where we treated this identification explicitly.]

- f. Using the mode functions that appear in the mode expansions (6) and (8), and the formula (5) for the Klein-Gordon inner product, compute explicit expressions for the Bogolubov coefficients $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$.
- g. Now take the limit where the curvature is weak, and can be treated as a linear perturbation. Deduce an expression for the total number of created particles in this limit. Later in the course we will derive expressions for the components of the stress energy tensor in this model, that describe the flux of created particles.