Physics 7683 : Problem Set 4

Due Thursday, Oct 29, 2009

1. *Change in field energy when quantum is created:* Consider a general static spacetime with metric $ds^2 = -e^{2\alpha(x^j)}dt^2 + h_{ij}(x^k)dx^i dx^j$. This metric has a timelike Killing vector field $\vec{\xi} = \partial/\partial t$, and associated conserved energy

$$\mathcal{E} = \int d^3x \sqrt{h} T_{ab} n^a \xi^b, \tag{1}$$

where the integral is over any spacelike hypersurface Σ, h_{ij} is the induced metric on Σ, $h = \det(h_{ij})$, and n^a is the unit, future directed normal to Σ. A free scalar field on this spacetime has a mode expansion $\hat{\Phi} = \sum_i [u_i \hat{a}_i + h.c.]$, where the mode functions u_i are purely positive frequency with respect to t.

Suppose that one starts in the vacuum state $|0\rangle$, and then one changes to a one-particle state $|\psi\rangle = \sum_i c_i \hat{a}_i^\dagger |0\rangle$, where $\sum_i |c_i|^2 = 1$ (as when an accelerated detector detects a particle from its bath of thermal radiation). In lecture we argued that the change in field energy due to this transition is

$$\langle\psi| \hat{\mathcal{E}} |\psi\rangle - \langle 0| \hat{\mathcal{E}} |0\rangle = \left\langle \mathcal{U}, i\frac{\partial}{\partial t}\mathcal{U} \right\rangle_{\rm KG}, \tag{2}$$

where $\mathcal{U} = \sum_i c_i u_i$ is the mode function associated with the one-particle state $|\psi\rangle$, and the inner product on the right hand side is the Klein-Gordon inner product. In this problem you will derive the formula (2) by integating the expected value of the stress energy tensor $\hat{T}_{ab}$ of the field.

We define the function $\mathcal{T}_{ab}$ of complex fields Φ and Ψ by $\mathcal{T}_{ab}(\Phi, \Psi) = \nabla_{(a}\Phi^* \nabla_{b)}\Psi - g_{ab}\nabla_c \Phi^* \nabla^c \Psi/2$. Then, the classical expression for the stress energy tensor of a Klein Gordon field Φ is just $\mathcal{T}_{ab}(\Phi, \Phi)$. For a quantum field, the stress energy tensor operator is given by

$$\hat{T}_{ab} =: \mathcal{T}_{ab}(\hat{\Phi}, \hat{\Phi}) : - \mathcal{S}_{ab}. \tag{3}$$

Here, in the first term on the right hand side, the complex conjugates in the definition of $\mathcal{T}_{ab}$ are interpreted to be Hermitian conjugages. The second term on the right hand side is a state-independent, c-number, piece that must be subtracted in order to regularize the operator. It is a nonlocal functional of the metric, and is the most difficult piece of the stress energy tensor to compute. We will discuss the computation of $\mathcal{S}_{ab}$ later in the course. In this problem we are interested in the difference between the expected values of $\hat{T}_{ab}$ in two difference states, which does not depend on $\mathcal{S}_{ab}$.

a. By inserting the mode expansion of the field into the definition (3), show that the change in the stress energy tensor is given by $\langle\psi| \hat{T}_{ab} |\psi\rangle - \langle 0| \hat{T}_{ab} |0\rangle = 2\mathcal{T}_{ab}(\mathcal{U}, \mathcal{U})$.

b. Use this result together with the defintion (1) of the conserved energy to show that the change in energy is

$$\delta\mathcal{E} = \langle\psi|\,\hat{\mathcal{E}}\,|\psi\rangle - \langle 0|\,\hat{\mathcal{E}}\,|0\rangle = \int_\Sigma d^3x\sqrt{h}\left[e^{-\alpha}\mathcal{U}^*_{,t}\mathcal{U}_{,t} + e^{\alpha}h^{ij}\mathcal{U}^*_{,i}\mathcal{U}_{,j}\right]. \tag{4}$$

c. Next, write down the Klein-Gordon equation that is satisfied by the mode function $\mathcal{U}$, integrate over all of space with a suitable choice of weighting function, and integrate by parts. Thereby derive the identity

$$\int_\Sigma d^3x\sqrt{h}\left[e^{-\alpha}\mathcal{U}^*\mathcal{U}_{,tt} + e^{\alpha}h^{ij}\mathcal{U}^*_{,i}\mathcal{U}_{,j}\right] = 0. \tag{5}$$

d. Now by combining the results (4) and (5) together with the definition of the Klein Gorder product, deduce the formula (2).

2. *Absorbtion of a Quantum from a Thermal Bath:* A harmonic oscillator with frequency ω is in a thermal state, so that the probability that the oscillator is in a state with energy $E = n\omega$ is $p_n \propto \exp[-\beta\omega n]$, and the expected value of the oscillators energy is $\langle E\rangle_\beta = \omega/(e^{\beta\omega} - 1)$. Now a "detector" is briefly brought into contact with the oscillator. With probability $n\epsilon$ (to lowest order in $\epsilon \ll 1$), the detector lowers the energy of the oscillator by ω, and with probability $1 - n\epsilon$, it leaves the state of the oscillator unchanged.

a. Fill in the details of the derivation sketched in lecture of a formula for the expected energy $\langle E\rangle$ of the oscillator after the interaction, assuming that the detector does succeed in removing an energy ω from the oscillator.

b. How, on the average, is the expected energy $\langle E\rangle$ of the oscillator changed by the detector, to leading order in ϵ? Take into account that the detector may or may not succeed in removing energy from the oscillator.