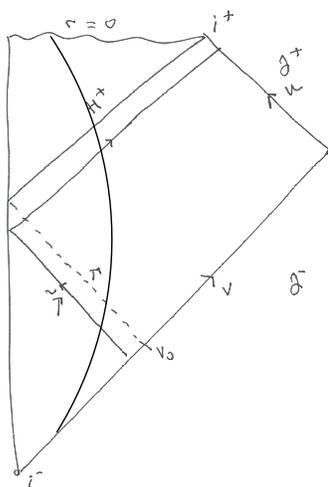


Physics 7683 : Problem Set 5

Due Thursday, Nov 12, 2009

1. *Near-horizon skimming null geodesics in a gravitational collapse spacetime:* Consider a spherically symmetric spacetime formed by gravitational collapse. Outside the collapsing matter (to the right of the curved arc in the figure), the geometry is Schwarzschild, which we describe with null coordinates $u = t - r_*$, $v = t + r_*$. We denote by v_0 the value of the v coordinate for the incoming radial null geodesic Γ that leaves past null infinity, reflects off the origin, and forms the future event horizon. We are interested in radial null geodesics $\tilde{\Gamma}$ that leave past null infinity just before Γ . These geodesics reflect off the origin, and then spend a long time in the vicinity of the horizon before escaping to future null infinity. Wavepacket modes that move along geodesics like $\tilde{\Gamma}$ are the key modes that are involved in the creation of outgoing Hawking quanta.



- a. In any two dimensional spacetime with metric $ds^2 = -e^{2\sigma(u,v)} du dv$, show that the vector fields $e^{-2\sigma}\partial_u$ and $e^{-2\sigma}\partial_v$ are geodesic.
- b. Let \vec{k} be the tangent vector to the geodesic Γ , and let \vec{l} be a null vector which is parallel transported along Γ , with $\vec{k} \cdot \vec{l} = -1$. Show that if the geodesics Γ and $\tilde{\Gamma}$ are close to one another, then the distance between the two geodesics is conserved, where distance is measured in terms of affine parameter separation ζ along a outgoing null geodesic that intersects Γ and $\tilde{\Gamma}$, with the normalization of the affine parameter ζ being fixed by $\partial/\partial\zeta = \vec{l}$ on Γ . [*Hint:* this can be derived either from the geodesic deviation equation or from part a.]
- c. Use part b. to argue that the interval Δv between Γ and $\tilde{\Gamma}$ at past null infinity is related to the interval ΔU between Γ and $\tilde{\Gamma}$ along an ingoing radial null geodesic \mathcal{C} outside the collapsing matter just after the black hole forms by $\Delta v = a\Delta U$, for some constant a .
- d. From part c. deduce that the value of the coordinate u at which the geodesic $\tilde{\Gamma}$ reaches future null infinity is given by

$$u = u(v) = -4M \ln \left[\frac{v_0 - v}{K} \right] + O(v_0 - v),$$

for some positive constant K .

2. Effective potential of a Black Hole: Show that the Klein-Gordon equation

$$\nabla_a \nabla^a \Phi - m^2 \Phi = 0$$

in the $r > 2M$ region of the Schwarzschild geometry has solutions of the form

$$\Phi(t, r, \theta, \phi) = R_l(r) \frac{e^{-i\omega t}}{r} Y_{lm}(\theta, \phi),$$

where

$$\left[-\frac{\partial^2}{\partial r_*^2} - \omega^2 + V_l(r_*) \right] R_l = 0,$$

and find the effective potential $V_l(r_*)$. Here r_* is the tortoise coordinate $r_* = r + 2M \ln(r/2M - 1)$.

3. Wavepacket mode basis: Consider a two dimensional flat spacetime with metric $ds^2 = -dudv$. An orthonormal basis for the set of leftward propagating modes is

$$u_\omega(v) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega v},$$

for $\omega > 0$. These satisfy $\langle u_\omega, u_{\omega'} \rangle = \delta(\omega - \omega')$, where the brackets denote the Klein Gordon inner product on the leftward propagating sector (see problem 2 of HW3). We define the wavepacket mode basis by

$$u_{j,n}(v) = \frac{1}{\sqrt{\Delta\omega}} \int_{j\Delta\omega}^{(j+1)\Delta\omega} d\omega \exp[2\pi i n \omega / \Delta\omega] u_\omega(v),$$

where $\Delta\omega > 0$ is a fixed parameter, and j and n are integers with $j > 0$ and $-\infty < n < \infty$.

- Show that the wavepacket mode $u_{j,n}$ is localized in v at $v \sim v_n = 2\pi n / \Delta\omega$ with a width of order $\Delta v \sim 2\pi / \Delta\omega$.
- Show that the modes are orthonormal, i.e.

$$\langle u_{j,n}, u_{j',n'} \rangle = \delta_{j,j'} \delta_{n,n'}.$$

- Show that the basis of wavepacket modes is complete.