

# Physics 7683 : Problem Set 6

*Due Thursday, Nov 26, 2009*

**1. Properties of Hawking Radiation at Future Null Infinity:** In lecture we derived the following formula for the Unruh state in maximally extended Schwarzschild spacetime, which we argued coincided near future near infinity with the asymptotic (late time) state produced in a spherically symmetric gravitational collapse:

$$|0, \text{U}\rangle = \prod_{\omega} \frac{1}{N_{\omega}} \exp \left\{ e^{-\beta\omega/2} \left[ t_{\omega} (\hat{a}_{\omega}^{\mathcal{I}^+})^{\dagger} + r_{\omega} (\hat{a}_{\omega}^{\mathcal{H}^+})^{\dagger} \right] (\hat{a}_{\omega}^{\text{II}, \mathcal{H}^-})^{\dagger} \right\} |0, \text{out}\rangle .$$

Here  $\beta = 8\pi M$  is the inverse temperature,  $t_{\omega}$  and  $r_{\omega}$  are the transmission and reflection coefficients for the effective potential, and  $N_{\omega} = (1 - e^{-\beta\omega})^{-1/2}$ . Also  $\mathcal{I}^+$  is future null infinity,  $\mathcal{H}^+$  is the future event horizon, and  $\mathcal{H}^-$  is the past event horizon. This formula applies for fixed values of the angular momentum quantum numbers  $l$  and  $m$ ; the full state is a tensor product over all  $l, m$ .

- a. Take the trace over the Hilbert space of modes  $u_{\omega}^{\text{II}, \mathcal{H}^-}$  on the past event horizon in region II, and then over the Hilbert space of modes on the future event horizon  $\mathcal{H}^+$ . Thereby derive the following density matrix in the Hilbert space of modes at future null infinity:

$$\hat{\rho}_{\mathcal{I}^+} = \prod_{\omega} \sum_{n_{\omega}=0}^{\infty} p_{n_{\omega}, \omega} |n_{\omega}, \mathcal{I}^+\rangle \langle n_{\omega}, \mathcal{I}^+| ,$$

with

$$p_{n_{\omega}, \omega} = (1 - e^{-\beta\omega}) \frac{[e^{-\beta\omega} |t_{\omega}|^2]^{n_{\omega}}}{[1 - |r_{\omega}|^2 e^{-\beta\omega}]^{n_{\omega}+1}} .$$

- b. Show that the mean occupation number of the mode  $\omega$  at  $\mathcal{I}^+$  is

$$|t_{\omega}|^2 \frac{1}{e^{\beta\omega} - 1} .$$

Give a physical interpretation of this formula.

**2. Radiation From An Accelerating Mirror as an Analog of Hawking Radiation:** Consider a free, massless scalar quantum field  $\Phi$  in a flat, two dimensional spacetime with coordinates  $(t, x)$  and metric  $ds^2 = -dt^2 + dx^2$ . Suppose that a mirror is moving in this spacetime, with location given by  $x = x_M(t)$ . Assume that the mirror is perfectly reflecting, and enforces the boundary condition  $\Phi(t, x_M(t)) = 0$ . The scalar field is defined to vanish to the left of the mirror; we consider only the region to the right of the mirror.

Suppose that the incoming (leftward propagating) state from past null infinity is the vacuum state. Compute the state at future null infinity (rightward propagating modes) for a given specified motion of the mirror. Show that for an appropriate choice of motion of the mirror, this state is thermal.

**3. Zero Angular Momentum Observers (ZAMOs) near Kerr Black Holes:** In lecture we wrote the Kerr metric in Boyer-Lindquist coordinates  $(t, r, \theta, \phi)$  in the form

$$ds^2 = -\alpha^2 dt^2 + h_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where the lapse function  $\alpha$ , the shift vector  $\beta^i$  and the spatial metric  $h_{ij}$  all depend only on  $r$  and  $\theta$ . We defined the ZAMOs to be observers who followed the worldlines given by

$$\frac{dx^i}{dt} = -\beta^i.$$

- Show that the worldlines of the ZAMOs are orthogonal to the  $t = \text{constant}$  surfaces.
- Show that the ZAMOs perceive the geometry to be unchanging, by showing that the worldline of a ZAMO coincides with the integral curve of some Killing vector field of the Kerr geometry.
- Show that near the horizon, the angular velocity of a ZAMO becomes

$$\Omega_{\text{ZAMO}} \rightarrow \Omega_{\text{H}} = \frac{a}{2Mr_{\text{H}}},$$

where  $a$  is the spin parameter of the black hole and  $r_{\text{H}} = M + \sqrt{M^2 - a^2}$  is the location of the event horizon.

- Show that the 4-acceleration of a ZAMO is given by  $\vec{a}_{\text{ZAMO}} = \nabla \ln \alpha$ . [Hint: this calculation is easiest in a coordinate system which rotates uniformly with respect to Boyer-Lindquist coordinates, in which the ZAMO is stationary.] Deduce that the magnitude  $a_{\text{ZAMO}}$  of the 4-acceleration near the event horizon diverges as

$$a_{\text{ZAMO}} \simeq \frac{\kappa}{\alpha},$$

where  $\kappa$  (the surface gravity) is a constant over the horizon. Show that  $\kappa = (r_{\text{H}} - M)/(2Mr_{\text{H}})$ .

**4. Scalar Field in an Expanding Universe:** Suppose that  $\Phi$  is a free, massless scalar field with curvature coupling  $\xi$ , with action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2}(\nabla\Phi)^2 - \frac{1}{2}\xi R\Phi^2 \right].$$

Suppose that  $\Phi$  propagates in a spatially flat, Friedman-Robertson-Walker cosmology with metric  $ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2)$ . Defining  $\varphi(\eta, \mathbf{x}) = \Phi(\eta, \mathbf{x})a(\eta)$ , show that the action can be written as the action of a massless scalar field in flat spacetime, but with a time-dependent mass:

$$S = \int d\eta \int d^3x \left[ \frac{1}{2}\varphi_{,\eta}^2 - \frac{1}{2}\delta^{ij}\varphi_{,i}\varphi_{,j} - \frac{1}{2}m(\eta)^2\varphi^2 \right],$$

where

$$m(\eta)^2 = -\frac{a_{,\eta\eta}}{a}(1 - 6\xi).$$

Note that this vanishes for the case of conformal coupling,  $\xi = 1/6$ .